A New Contraction-Based NMPC Formulation

Without Stability-Related Terminal Constraints

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Recalls on stability assessment in NMPC

Mayne et al. Automatica 2000

$$\min_{\mathbf{u}\in\mathbb{U}(x)} \left[F(\mathbf{x}_q^{\mathbf{u}}(x)) + \sum_{\ell=1}^{q} L(\mathbf{x}_\ell^{\mathbf{u}}(x), \mathbf{u}_\ell)\right] \quad \text{under} \quad \mathbf{x}_q^{\mathbf{u}} \in \mathbb{X}_F$$

Stability-related ingredients

Main assumption in the stability proof

F is a CLF for some feedback $\kappa(\cdot)$ defined on $\bar{\mathbb{X}}_F$

Typical features

- $\mathbb{X}_F$ is a *small set around the target*
- ($\mathbf{x}_q^{\mathbf{u}} \in \mathbb{X}_F$) Needs long prediction horizons q
- Computationally expensive ($\rightarrow$ almost never used by practitioners)

Recalls on stability assessment in NMPC

Grüne et al. JOTA 2010

$$\min_{\mathbf{u}\in\mathbb{U}(x)} \left[\qquad + \sum_{\ell=1}^{q} L(\mathbf{x}_\ell^{\mathbf{u}}(x), \mathbf{u}_\ell) \right]$$

$q \to \infty$

Main assumption in the stability proof

q is sufficiently high

Typical features

- Long prediction horizon
- Analysis of sub-optimality
- Generally not constructive results

This contribution

Question

Are there **provably stable** NMPC formulations with

1. Short prediction horizon
2. No stability-related terminal constraints

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Next →

- ✓ Contraction property
- ✓ Existing formulations
- ✓ New formulation

Illustrative example

Dynamic system

$$\dot{x}_1 = x_2 \ ; \ \dot{x}_2 = x_1 u$$

candidate Lyapunov function

$$V(x) = \frac{1}{2}\left[x_1^2 + x_2^2\right]$$

Compute $\dot{V}$

$$\dot{V}(x) = x_1 x_2 (1 + u)$$

For classical point-wise design

$x_1 x_2 = 0$ is a singular surface

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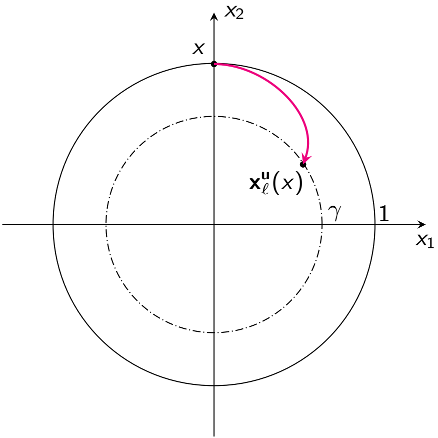

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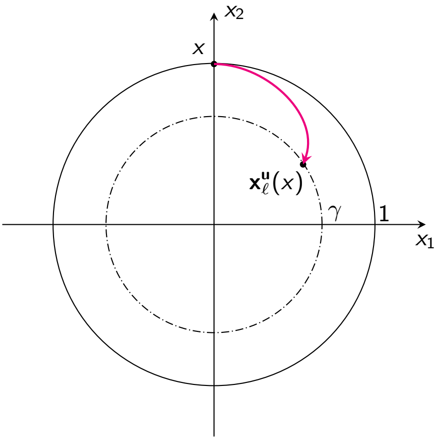

$\forall x \in \mathbb{G}, \; \exists \mathbf{u} \in \mathbb{U}^N$ s.t.

$$\min_{\ell=1}^{N}\Big[V(\mathbf{x}_\ell^{\mathbf{u}}(x))\Big] \leq \gamma V(x)$$

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More generally, under mild stabilizability assumptions, ANY positive definite V meets the contraction property (with an appropriate, rather short N)

MA, Springer, 2006 / Bobiti & Lazar 2014

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Intuitive *bad* formulation

$$\min_{\mathbf{u}\in\mathbb{U}(x_k)} \left[\sum_{\ell=1}^{N} L(\mathbf{x}^{\mathbf{u}}_{\ell}(x_k),\mathbf{u}_{\ell})\right] \quad \text{under} \quad V(\mathbf{x}^{\mathbf{u}}_{N}(x_k)) \leq \gamma V(x_k)$$

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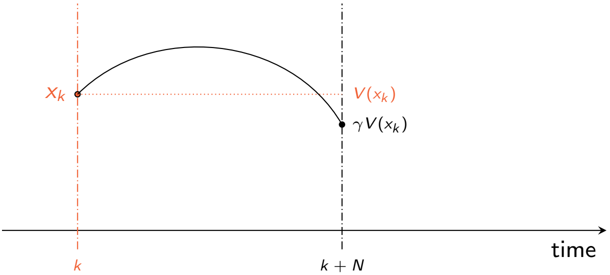

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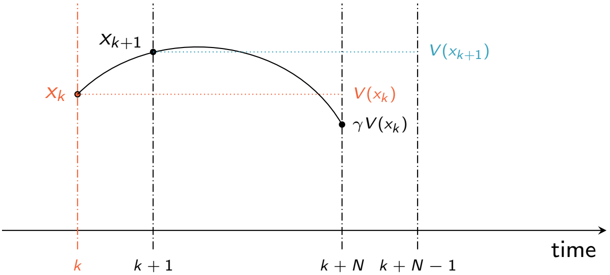

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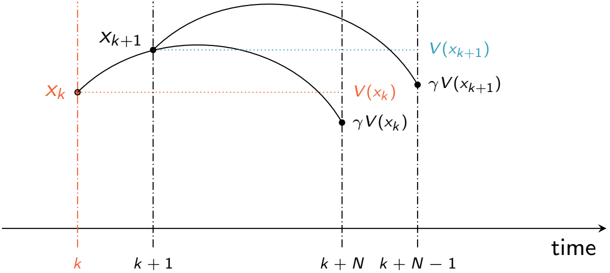

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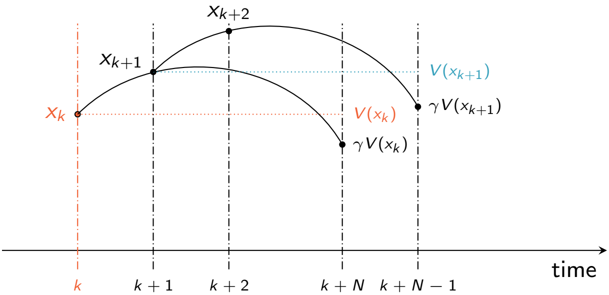

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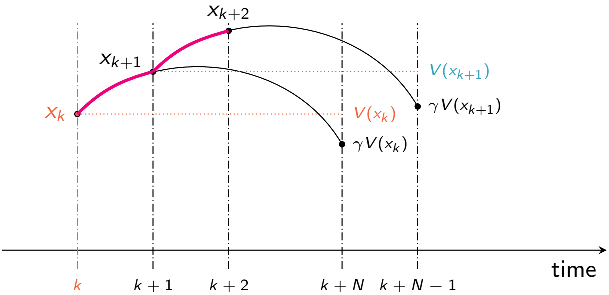

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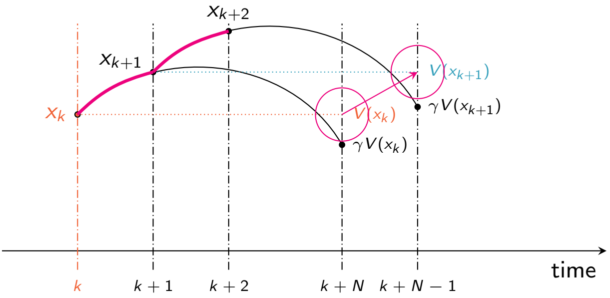

Kothare et al. IEEE TAC 2000 (Formulation 1)

As long as $V(x_{k_i+j}) > \gamma V(x_{k_i})$,

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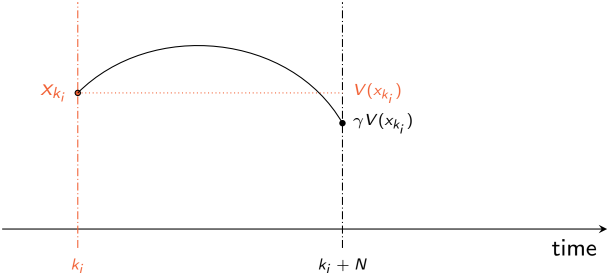

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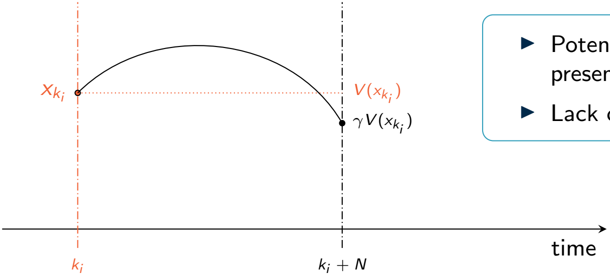

- Potential unfeasibility in presence of disturbance
- Lack of reactivity

Kothare et al. IEEE TAC 2000 (Formulation 2)

Apply the optimal solution $\mathbf{u}^*(x_{k_i})$ in open-loop until contraction occurs at some instant $k_i + \Delta^*(x_{k_i})$

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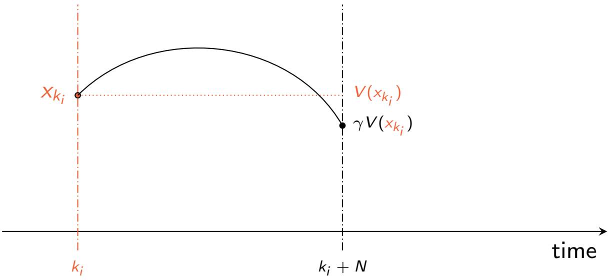

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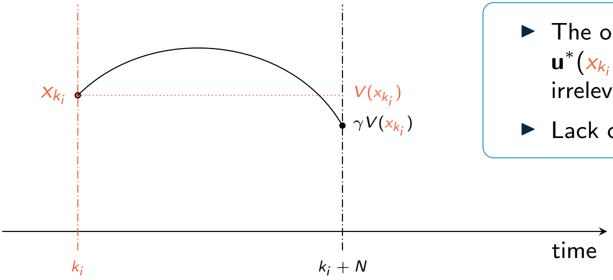

- The optimal solution $\mathbf{u}^*(x_{k_i})$ may become irrelevant
- Lack of reactivity

MA, NMPC Workshop 2005

- Take V **satisfying the contraction property** for some N
- Use the prediction horizon as a decision variable
- Penalize the maximum excursion

$$\min_{(\ell,\mathbf{u})\in\{1,\dots,N\}\times\mathbb{U}}\left[V(\mathbf{x}^{\mathbf{u}}_{\ell}(x_k)) + \alpha\frac{\ell}{N}\min\left\{\varepsilon, \|V(\mathbf{x}^{\mathbf{u}}_{1:\ell}(x_k))\|^2_{\infty}\right\}\right]$$

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✓ Do not explicitly use any stability-related constraints

✗ **Same V for performance and stability**

✗ Existence result ($\exists$ sufficiently small (α,ε) ...)

Definitions & Assumptions

Problem's data

System

$$x_{k+1} = f(x_k, u_k)\ ,\ \ u_k \in \mathbb{U}$$

State constraints

$$\mathbb{G} := \{x \mid g(x) \leq 0\}$$

Stage cost

$$\Phi(x, \mathbf{u}, q) := \sum_{\ell=1}^{q} L(\mathbf{x}_\ell^{\mathbf{u}}(x), \mathbf{u}_\ell)$$

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Assumption 1

$\exists \bar{L} > 0$ s.t.

$$\forall (x, u) \in \mathbb{G} \times \mathbb{U}, \quad 0 \leq L(x, u) \leq \bar{L}$$

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Assumption 2

$\exists$ positive definite function V, a contraction factor $\gamma \in (0, 1)$ and a prediction horizon N such that $\forall x \in \mathbb{G}$, $\exists \mathbf{u} \in \mathbb{U}^N$ such that:

$$\forall \ell \in \{1, \dots, N\} \qquad \mathbf{x}_\ell^{\mathbf{u}}(x) \in \mathbb{G}$$

$$\underline{V}(x, \mathbf{u}, N) := \min_{\ell=1}^{N} \Big[V(\mathbf{x}_\ell^{\mathbf{u}}(x)) \Big] \leq \gamma V(x)$$

Main Result

Optimization problem $\mathcal{P}(x,z)$

$$\mathbf{u}^*(x,z) \leftarrow \min_{(\mathbf{u},q)} \Big[J^{(x,z)}(\mathbf{u},q)\Big] := z\cdot\Phi(x,\mathbf{u},q) + \alpha \min_{\ell=1}^{q}\Big[V(\mathbf{x}_\ell^{\mathbf{u}}(x))\Big]$$

(Physical state constraints) under $\mathbf{x}_\ell^{\mathbf{u}}(x) \in \mathbb{G} \quad \forall \ell \in \{1,\dots,q\}$

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NMPC Feedback Law

$$z^+ = \begin{cases} z & \text{if } V(x) > z \\ \beta z & \text{if } V(x) \le z \end{cases}$$

$$K_{MPC}(x,z) := \mathbf{u}^*_1(x,z)$$

for some arbitrary $\beta \in (0,1)$.

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$$\alpha > \frac{2N\bar{L}}{1-\gamma}$$

then $x = 0$ is asymptotically stable for all $(x_0, z_0) \in \mathbb{G} \times \mathbb{R}_+^*$

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(No need to solve optimization problem with integer decision variable)

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2) This formulation enables a provably **Globally** stable formulation despite of **saturation** on the control input.

(Generally impossible with standard MPC formulations with fixed terminal set).