

A low dimensional contractive NMPC scheme for nonlinear systems stabilization

Theoretical Framework & Numerical investigation on relatively
fast systems

Mazen Alamir ¹

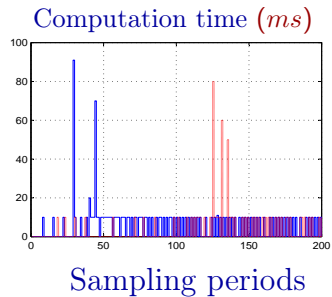
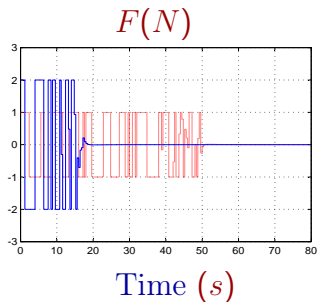
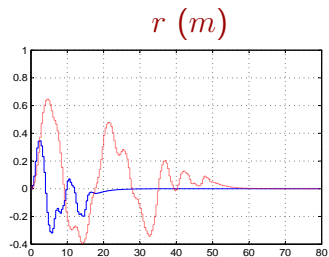
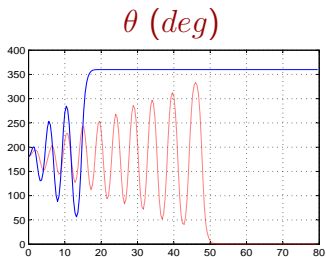
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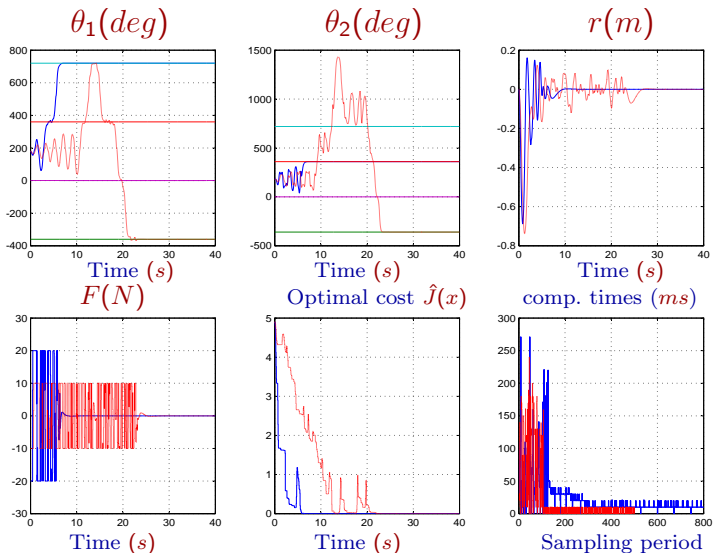
NMPC'05, Germany

Motivation

- ▶ Classical formulations are "**Optimality oriented**"
- ▶ Stability is achieved by auxiliary "*tricks*"
 - ▶ Final equality constraint
 - ▶ Final inequality constraint / terminal region
 - ▶ Infinite prediction horizon
- ▶ The resulting problem is heavy to tackle and formulate numerically whatever is the implementation "*(complete / real time)*"

What if the stability was the main issue ... ?





A new contractive formulation

- ▶ Finite & **short** prediction horizon
- ▶ **Without** any final or terminal requirements
- ▶ **provably stable** closed-loop
- ▶ **Low dimensional** decision variable (scalar & 2-dimensional)
- ▶ Solution uses only **black-box** simulator.

Outline

THE BASIC IDEA : THE CONTRACTION PROPERTY

USING THE CONTRACTION PROPERTY IN FEEDBACK DESIGN

An intuitive bad formulation

Classical contractive Receding horizon schemes

The proposed contractive formulation

The class of systems

The open-loop control parametrization

The contraction property

The new contractive RH formulation

Illustrative examples

The simple inverted pendulum : A self contained RH control

The double inverted pendulum : a hybrid control scheme

Conclusion

A PRELIMINARY EXAMPLE

Consider the nonlinear system

$$\dot{x}_1 = x_2 \quad ; \quad \dot{x}_2 = x_1 u$$

and the "*candidate*" Lyapunov function

$$V(x) = \frac{1}{2} [x_1^2 + x_2^2]$$

Compute the derivative of V

$$\dot{V}(x) = x_1 x_2 (1 + u)$$

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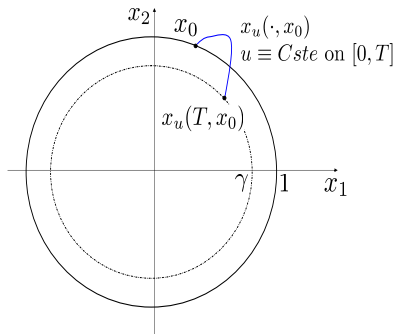
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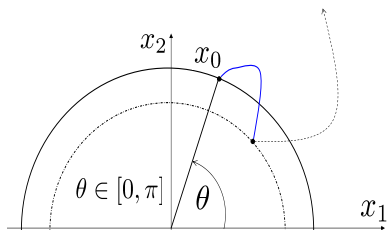
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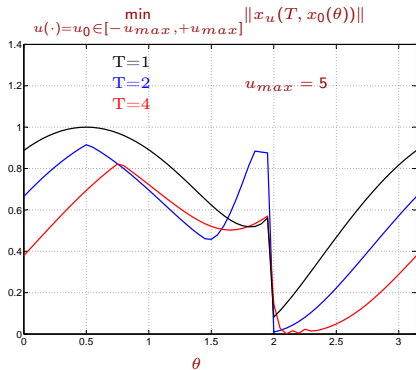
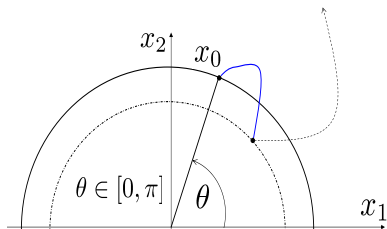
But...



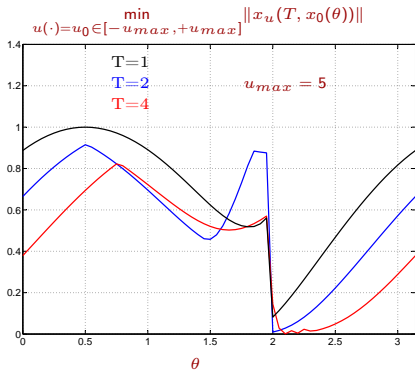
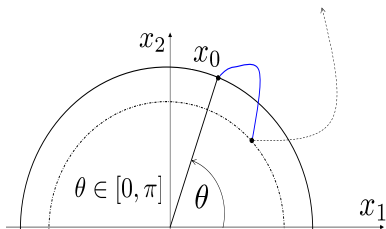
$$\min_{u(\cdot) \equiv u_0 \in [-u_{max}, +u_{max}]} \|x_u(T, x_0(\theta))\|$$



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To summarize : A long term contraction property

Whatever is the initial state $x_0 \in B(0,1)$, there exists constant control $u \in [-5, +5]$ such that

$$\|V(x_u(2, x_0))\| \leq 0.9 V(x_0) \quad \text{or} \quad \|V(x_u(4, x_0))\| \leq 0.82 V(x_0)$$

A bad formulation

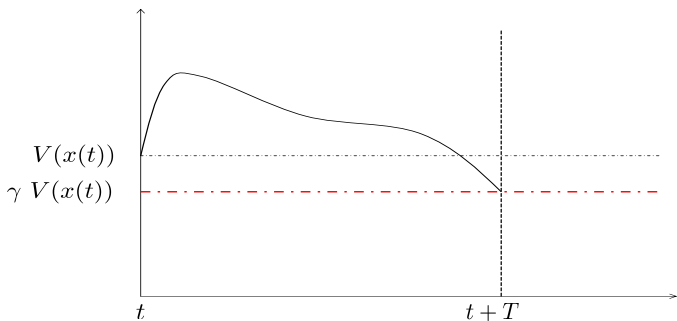
Define a receding horizon feedback based on the following open-loop optimization problem

$$\min_{\mathbf{u}(\cdot), \Delta \in [0, T]} \int_0^{\Delta} L(x_u(\tau, x(t))) d\tau \quad \text{under } V(x_u(t + \Delta, x(t))) \leq \gamma V(x(t))$$

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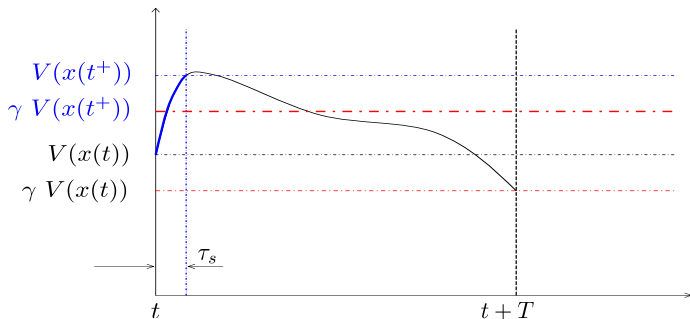
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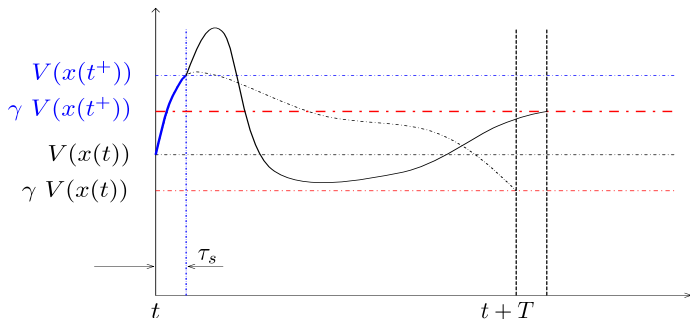
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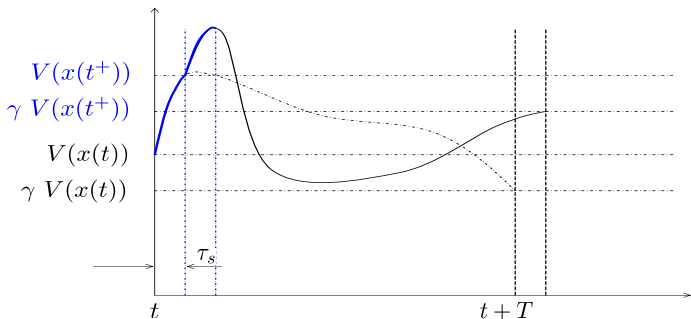
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Updating systematically the contractive constraint

$$V(x_u(\Delta, x(t))) \leq \gamma V(x(t))$$

May cause instability in closed-loop

Kothare, S. L. de Oliveira and Morari, M. IEEE-TAC Vol 45 pp 1053-1071 (2000)

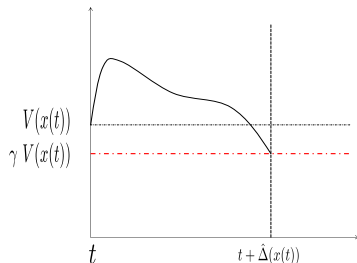
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Either

Use the open-loop control

$$\hat{u}(\cdot, x(t))$$

on $[t, t + \hat{\Delta}(x(t))]$



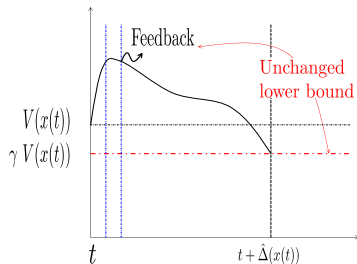
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Or

Memorize $x(t)$ and use

$$V(x_u(\Delta, x(t + k\tau_s))) \leq \gamma V(x(t))$$

in a RH scheme during the time interval $[t, t + \hat{\Delta}(x(t))]$



Consider nonlinear systems

$$\dot{x} = f(x, u) \quad ; \quad x \in \mathbb{R}^n \quad ; \quad u \in \mathbb{R}^m \quad ; \quad f \text{ continuous}$$

satisfying the following assumption

Infinitely fast state excursions need infinite control

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Infinitely fast state excursions need infinite control

For all finite horizon $T > 0$,

$$\lim_{\|x_0\| \rightarrow \infty} \left[\min_{\mathbf{u} \in \mathbb{W}^{[0,T]}} \min_{t \in [0,T]} \|F(t, x_0, \mathbf{u})\| \right] = \infty$$

for all compact subset $\mathbb{W} \subset \mathbb{R}^m$.

b

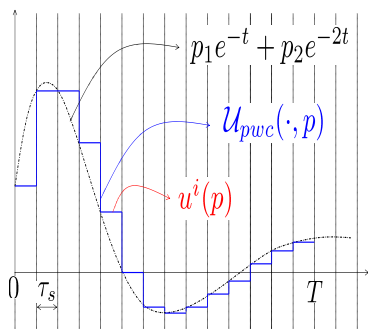
- ✓ Choose a sampling period τ_s
- ✓ Define a τ_s -piece-wise constant control profile

$$\mathcal{U}_{pwc}(\cdot, p) \quad ; \quad p \in \mathbb{P}$$

- ✓ The parametrization is called "*translatable*" if for all $p \in \mathbb{P}$, there is $p^+ \in \mathbb{P}$ s.t.

$$u^i(p^+) = u^{i+1}(p)$$

$$\forall i \in \{1, \dots, N-1\}$$



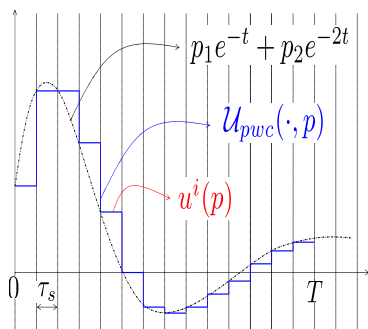
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$$p^+ = \begin{pmatrix} e^{-\tau_s} & 0 \\ 0 & e^{-2\tau_s} \end{pmatrix} p$$

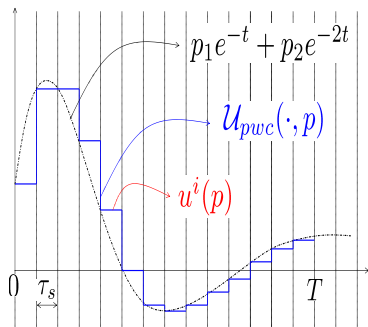
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Notation $F(\cdot, x, p)$, $V(\cdot, x, p)$

$$p^+ = \begin{pmatrix} e^{-\tau_s} & 0 \\ 0 & e^{-2\tau_s} \end{pmatrix} p$$

The strong contraction property

1. $\exists \gamma \in]0, 1[$ s.t. for all x , there exists $p^c(x) \in \mathbb{P}$ such that

$$\min_{q \in \{1, \dots, N\}} V(q\tau_s, x, p^c(x)) \leq \gamma V(x)$$

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2. $p_c(\cdot)$ is bounded over bounded sets
3. \exists a continuous function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ s.t. for all x :

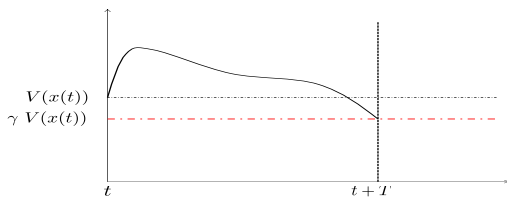
$$\|V_{1 \rightarrow N}(\cdot, x, p^c(x))\|_\infty \leq \varphi(x) \cdot V(x)$$

where

$$\|V_{1 \rightarrow q}(\cdot, x, p)\|_\infty = \max_{i \in \{1, \dots, q\}} V(i\tau_s, x, p)$$

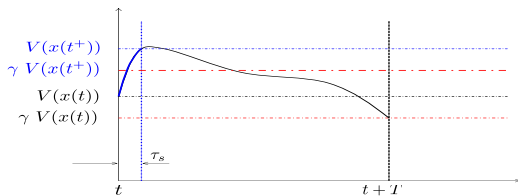
The proposed contractive formulation

The new contractive RH formulation



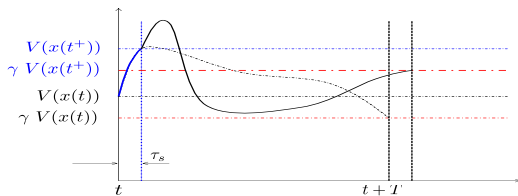
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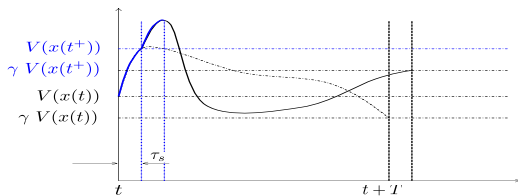
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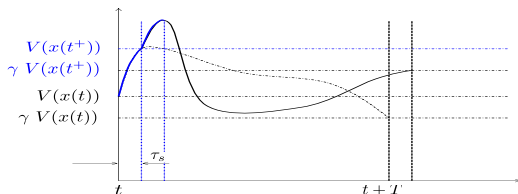
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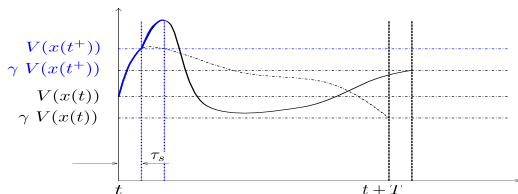


The open-loop optimal control problem

$$\min_{(q,p) \in \{1, \dots, N\} \times \mathbb{P}_{\mathbb{X}}} V(q\tau_s, x, p) + \alpha \frac{q}{N} \cdot \min \left\{ \varepsilon^2, \|V_{1 \rightarrow q}(\cdot, x, p)\|_{\infty}^2 \right\}$$

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The receding-horizon state feedback

$$u(k\tau_s + \tau) = u^1(\hat{p}(x(k\tau_s))) \quad \forall \tau \in [0, \tau_s]$$

Basic Result

If the following conditions hold

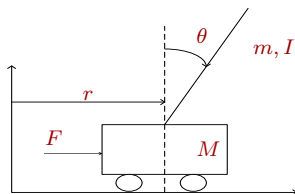
1. Continuity (system/parametrization)
2. Infinitely fast excursions need infinite controls
3. The control parametrization is translatable on

$$\mathbb{P}_{\mathbb{X}} := \mathbb{P} \cap B\left(0, \sup_{x \in \bar{B}(0, \rho(\mathbb{X}))} \|p^c(x)\| + \varepsilon_0\right) \subseteq \mathbb{P} \subseteq \mathbb{R}^{n_p}$$

Then, \exists sufficiently small $\varepsilon > 0$ and $\alpha > 0$ such that the RH feedback is well defined and makes the origin $x = 0$ asymptotically stable for the resulting CL dynamics with a region of attraction that contains \mathbb{X} .

Illustrative examples

The simple inverted pendulum : A self contained RH control



The system equations

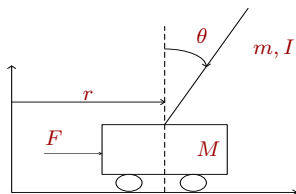
$$\begin{pmatrix} mL^2 + I & mL \cos \theta \\ mL \cos \theta & m + M \end{pmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{r} \end{pmatrix} = \begin{pmatrix} mLg \sin \theta - k_\theta \dot{\theta} \\ F + mL\dot{\theta}^2 \sin \theta - k_x \dot{r} \end{pmatrix}$$

A pre-compensator

$$F = -K_{pre} \begin{pmatrix} r \\ \dot{r} \end{pmatrix} + u$$

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The simple inverted pendulum : A self contained RH control



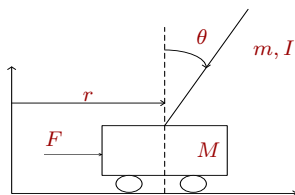
The system equations

$$\dot{x}_1 = x_3 \quad ; \quad \dot{x}_2 = x_4$$

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = [M(x)]^{-1} \begin{pmatrix} mLg \sin(x_1) - k_\theta \cdot x_3 \\ -K_{pre_1} x_2 - K_{pre_2} x_4 + mLx_3^2 \sin(x_1) - k_x x_4 + u \end{pmatrix}$$

Illustrative examples

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Control parametrization

$$u^i(p) = p \cdot e^{-t_i/t_r} \quad ; \quad t_i = \frac{(i-1)\tau_s}{N}$$

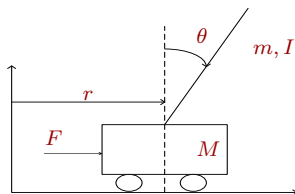
where $p \in \mathbb{P}(x) := [p_{min}(x), p_{max}(x)]$ s.t

$$p_{min}(x) = -F_{max} + K_{pre_1}x_2 + K_{pre_2}x_4$$

$$p_{max}(x) = +F_{max} + K_{pre_1}x_2 + K_{pre_2}x_4$$

Illustrative examples

The simple inverted pendulum : A self contained RH control



Use the contractive RH formulation given by :

$$\begin{aligned}
 V(x) &= \frac{1}{2} \left[\dot{\theta}^2 + \beta r^2 + \dot{r}^2 \right] + [1 - \cos(\theta)]^2 \\
 \min_{(q,p) \in \{1, \dots, N\} \times \mathbb{P}(x)} & V(q\tau_s, x, p) + \frac{\alpha}{N} \cdot \min\{\varepsilon, \|V_{1 \rightarrow q}(\cdot, x, p)\|_{\infty}\} \\
 u(k\tau_s + \tau) &= u^1(\hat{p}(x(k\tau_s))) \quad \forall \tau \in [0, \tau_s]
 \end{aligned}$$

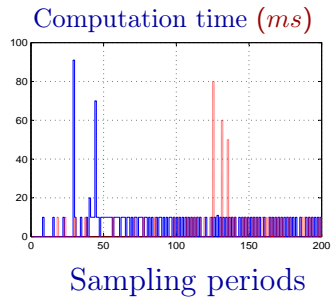
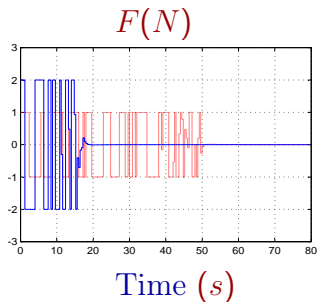
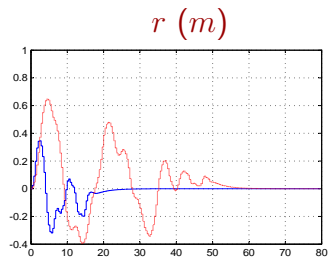
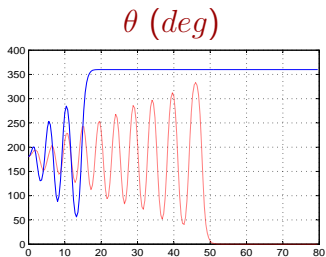
The parameters of the controller

parameter	value	signification
τ_s	0.4 s	sampling period
N	8	horizon length
t_r	0.2	Constant for the control param.
$\alpha = \varepsilon$	0.01	
K_{pre}	(2.5 10.0)	cost function parameters
F_{max}	$\in \{1.0, 2.0\}$	Pre-compensation gain
β	$\in 10$	saturation level on F
		weighting coefficient on r

Runs on a 1.3 GHz Pentium-III

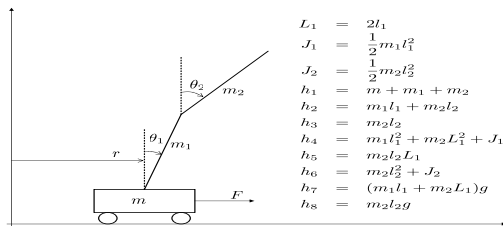
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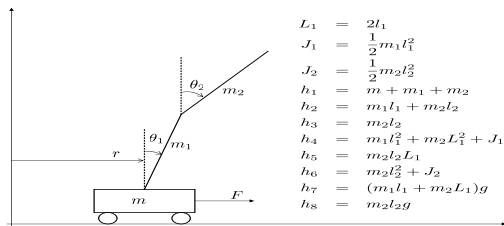
Illustrative examples

The double inverted pendulum : a hybrid control scheme



Illustrative examples

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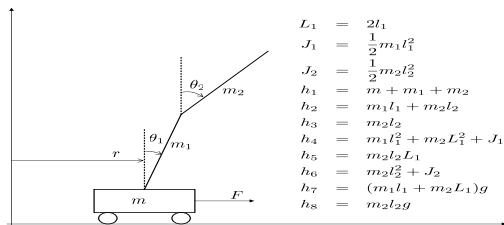


System equations

$$\begin{aligned}
 h_1\ddot{r} + h_2\ddot{\theta}_1 \cos \theta_1 + h_3\ddot{\theta}_2 \cos \theta_2 &= h_2\dot{\theta}_1^2 \sin \theta_1 + h_3\dot{\theta}_2^2 \sin \theta_2 + F \\
 h_2\ddot{r} \cos \theta_1 + h_4\ddot{\theta}_1 + h_5\ddot{\theta}_2 \cos(\theta_1 - \theta_2) &= h_7 \sin \theta_1 - h_5\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \\
 h_3\ddot{r} \cos \theta_2 + h_5\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + h_6\ddot{\theta}_2 &= h_5\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + h_8 \sin \theta_2
 \end{aligned}$$

Illustrative examples

The double inverted pendulum : a hybrid control scheme

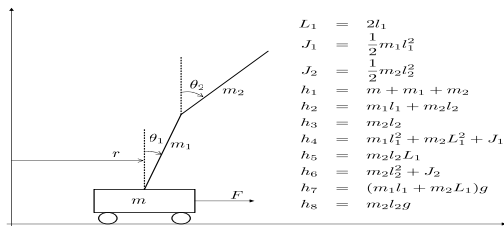


Pre-compensation

$$F = -K_{pre} \cdot \begin{pmatrix} r \\ \dot{r} \end{pmatrix} + u$$

Illustrative examples

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Control parametrization

$$u^i(p) = p_1 \cdot e^{\lambda_1 t_i} + p_2 e^{-\lambda_2 t_i} \quad ; \quad t_i = \frac{(i-1)\tau_s}{N}$$

$$p_{min}(x) := \frac{1}{2} \left[-F_{max} + K_{pre} \begin{pmatrix} r \\ \dot{r} \end{pmatrix} \right] \quad ; \quad p_{max}(x) := \frac{1}{2} \left[+F_{max} + K_{pre} \begin{pmatrix} r \\ \dot{r} \end{pmatrix} \right]$$

The contractive RH controller

$$\begin{aligned}
 V(x) &= \frac{h_4}{2} \dot{\theta}_1^2 + \frac{h_6}{2} \dot{\theta}_2^2 + h_5 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + h_7 [1 - \cos(\theta_1)] + \\
 &+ h_8 [1 - \cos(\theta_2)] + h_1 [r^2 + \dot{r}^2] \\
 u(k\tau_s + t) &= K_{RH}(x(k\tau_s)) := u^1(\hat{p}(x(k\tau_s))) \quad ; \quad t \in [0, \tau_s[
 \end{aligned}$$

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A local LQR controller

$$K_L(x) = -L \cdot \begin{pmatrix} x_1^m \\ x_2^m \\ x_3 \\ \vdots \\ x_6 \end{pmatrix}$$

solving the discrete time Riccati equation

$$A_d^T S A_d - S - (A_d^T S B_d)(R + B_d^T S B_d)^{-1}(B_d^T S A_d) + Q = 0$$

Hybrid controller for swing-up and stabilization of the double inverted pendulum

To summarize, the hybrid controller is given by

$$u(k\tau_s + \tau) = \begin{cases} K_{RH}(x(k\tau_s)) & \text{if } \|x(k\tau_s)\|_S^2 > \eta \\ K_L(x(k\tau_s)) & \text{otherwise} \end{cases}$$

Hybrid controller for swing-up and stabilization of the double inverted pendulum

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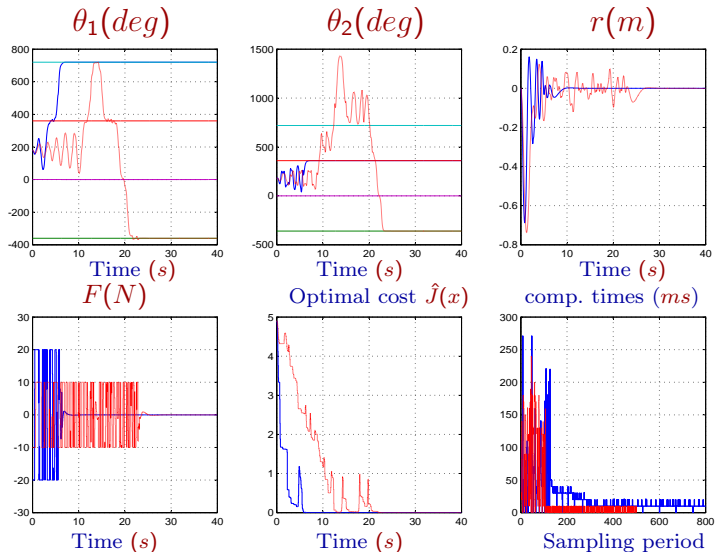
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The parameters of the controller

parameter	value	signification
τ_s	0.3 s	sampling period
N	10	horizon length
L	(360, 29)	(linear controller gain)
(λ_1, λ_2)	(100, 20)	Control parametrization
η	1.0	switching threshold
i_{max}	20	Max number of function evaluation

Illustrative examples

The double inverted pendulum : a hybrid control scheme



Conclusion

Whatever are the numerical tools in your hand ...

Don't forget that the added-value of a control-designer is

THE PROVABLY SAFE, SIMPLE & DEDICATED FORMULATION

Don't rely on a black-box tool to overcome a bad formulation,
...you would pay it by "*lines of code*" and even dollars in the final
product

The gains in efficiency are multiplicative