A State-Dependent Updating Period for Certified Real-Time Model Predictive Control

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***Abstract*—In this paper, a state-dependent control updating period strategy is proposed for use in interrupted implementation of real-time Model Predictive Control (MPC). The strategy can be used as soon as a certification bound is available for the underlying optimization algorithm. Moreover, a new fast-Gradient based certifiable algorithm is proposed with the associated certification bounds for convex generally constrained optimization problems.**

***Index Terms*—Certification, MPC, real-time, stability.**

I. Introduction

The use of Model Predictive Control (MPC) [6] to control systems with fast dynamics enhanced an intense research activity whose goal is to derive certification bounds on different optimization algorithms such as fast gradient [12], proximal Newton [9], accelerated dual gradient [10], or alternate minimization method [11], to cite but few examples. Regarding the other alternatives, active set iterations [5], while computationally efficient and while showing a provably finite number of iterations to converge (for QP problems), seem to resist to the derivation of convergence rates which makes impossible the computation of certification bounds. As for interior point methods [3], [4], [13], certification bounds exist [7] but seem to be systematically over pessimistic [12].

Certification bounds give the minimum number of iterations of a specific algorithm that can guarantee a prescribed level of precision on the cost function and on soft constraints violation. Such non vanishing precision levels are necessary since for fast systems, reasonable Control Updating Periods (**CUPs**), during which iterations take place, may not be long enough to get the exact optimal solution. Therefore, certification bounds come into play only once a precision level is defined but do not answer the question of how this level should be determined so that certified stability result can be derived.

More precisely, let us consider a dynamic system given by

$$\dot{x} = F(x,u) \quad \text{where} \quad (x,u) \in \mathbb{R}^n \times \mathbb{R}^{n_u} \tag{1}$$

in which u is the control while x is the extended-state vector gathering the physical state, the set-point value and the current estimated value of the disturbance. Therefore, in addition to the equations governing the evolution of the physical state, F includes evolution models for the set-point and the uncertainties that are used in the MPC-related prediction. In this prediction, the input control profile is supposed to be parametrized through a finite dimensional vector of decision variables p such that control profile over $[t, t+T]$ is given by:

$$u(t+s) := \mathcal{U}(s, p(t))\,; \quad p \in \mathbb{R}^{n_p} \tag{2}$$

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where $\mathcal{U}$ is some predefined map. Consider an MPC formulation that is defined by the following optimization problem:

$$\min_{p\in\mathbb{R}^{n_p}} [f_0(p,x)] \text{ under } c_i(p,x) \leq 0 \quad \forall i \in I_h \cup I_s \tag{3}$$

where the sets of indices I_h and I_s define a partition of the set $\{1,\ldots,n_c\}$ into hard and soft constraints respectively.

For any pair of strictly positive reals $\bar{\epsilon} := (\epsilon_0, \epsilon_\psi)$, p is called a $\bar{\epsilon}$-suboptimal solution to (3) for some x if p satisfies all the hard constraints, does not violate any soft constraint by more than ϵ_ψ and such that $|f_0(p,x) - f^{\text{opt}}(x)| \leq \epsilon_0$, where $f^{\text{opt}}(x)$ is an optimum of (3).

Assume that there exists a certifiable optimization algorithm A that can be used to solve (3). More precisely, for any $x(t)$, any given pair $(\epsilon_0, \epsilon_\psi)$ and any initial guess p_0, it is possible to determine the number of iterations $N(p_0, \epsilon_0, \epsilon_\psi, x(t))$ that lead to a $(\epsilon_0, \epsilon_\psi)$-suboptimal solution $\hat{p}^* = A(p_0, \epsilon_0, \epsilon_\psi, x(t))$ to (3). This suboptimal solution can be obtained provided that the CUP is larger than $\tau_u := \tau_c \times N(p_0, \epsilon_0, \epsilon_\psi, x(t))$, where τ_c is the time needed by the underlying hardware to perform a single iteration of the algorithm. Therefore, the computation should start at $t - \tau_u$ with a state $\hat{x}(t)$ that is predicted at $t - \tau_u$ with prediction error that increases with τ_u. Consequently, higher precision ϵ_0 leads to higher τ_u and hence, higher mismatch on the problem definition. This obviously reveals a trade-off for which the better choice ϵ_0 is likely to be dependent on the state $x(t)$, the computation time τ_c, the uncertainty level and the very definition of the MPC framework.

The derivation of a state-dependent targeted-precision framework that accommodates for the above coupling is the main contribution of the present technical note. Moreover, while this framework can be easily adapted to any certification bound meeting the above specifications, the presentation is based on a new certifiable fast-gradient based algorithm for convex problems with general constraints which is proposed in this technical note as a secondary result.

The technical note is organized as follows: in Section II, a new certifiable algorithm is proposed for a class of convex problems. Section III derives the main results regarding certifiable MPC framework. Finally, Section IV gives a general discussion and proposes some hints for further investigation.

Due to space limitation, all the proofs of Section II are given in [2] where many clarifying figures and examples are also provided.

II. A New Certifiable Algorithm

Let us consider optimization problems of the form

$$p^{\text{opt}} \leftarrow \min_{p\in\mathbb{R}^{n_p}} [f_0(p)] \text{ under } c_i(p) \leq 0 \quad \forall i \in I_h \cup I_s \tag{4}$$

which is an instantiation of (3) for a given state x. The certification bound derived in this section is based on a specific version of the certification bounds of the Fast Gradient (**FG**) algorithm [8]. Since the latter can be derived for unconstrained problems or for problems with only simple-box-like constraints, the problem (4) which is intended to include more general constraints is here approximately solved by solving the following unconstrained problem:

$$p^* \leftarrow \min_{p\in\mathbb{R}^{n_p}} [f_0(p) + \rho\psi(p)] =: f(p) \tag{5}$$

where $\psi(p)$ is given by

$$\psi(p) := \sum_{i\in I_s}[\max\{0, c_i(p)\}]^2 + \sum_{i\in I_h}[\max\{0, c_i(p)+\varepsilon_\psi\}]^2 \quad (6)$$

which is a standard penalty-based approach. Note that the definition of ψ is such that if $\psi(p)\le \epsilon_\psi^2$ then p satisfies all the hard constraints and does not violate any soft constraints by more than ϵ_ψ. Some notation and working assumptions needed to state and establish the certification result are given in the next section.

A. Definitions, Notation, and Working Assumptions

In what follows, p^{opt}, p^* denote the solutions of (4) and (5), respectively, while $\hat{p}^*$ is used to denote an approximate solution of (5) obtained after a finite number of iterations. $f'(p)$, $f_0'(p)$ and $\psi'(p)$ denote gradients w.r.t p. The euclidien norm of $f'(p)$ is denoted by $g(p)=\|f'(p)\|$. For a scalar continuously differentiable function ℓ defined on $\mathbb{R}^n$, the notation $\ell\in\mathcal{S}^1_\mu$ states that ℓ is a μ-strongly convex function, namely

$$\ell(p_2)\ge \ell(p_1)+\langle \ell'(p_1), p_2-p_1\rangle + \frac{\mu}{2}\|p_2-p_1\|^2. \quad (7)$$

Similarly, the notation $\ell\in\mathcal{F}^1_L$ indicates that the continuously differentiable function ℓ satisfies for all (p_1,p_2)

$$\ell(p_2)\le \ell(p_1)+\langle \ell'(p_1), p_2-p_1\rangle + \frac{L}{2}\|p_2-p_1\|^2. \quad (8)$$

When ℓ satisfies both (7) and (8), the notation $\ell\in\mathcal{S}^1_{\mu,L}$ is used. The set $\mathcal{C}$ denotes the set of singular points of $f(\cdot)$, namely the set of p such that $g(p)=0$. Given a subset $\mathcal{A}\subset\mathbb{R}^{n_p}$, the notation $d(p,\mathcal{A})$ refers to the distance from p to $\mathcal{A}$, namely $d(p,\mathcal{A}):=\min_{z\in\mathcal{A}}\|z-p\|$. The short notation $d(p):=d(p,\mathcal{C})$ is used for the specific set $\mathcal{C}$. The set $\mathcal{A}_{\psi=0}$ is the set of p such that $\psi(p)=0$. Given a bounded subset $\mathbb{P}$, $\delta_{\mathbb{P}}$ denotes the diameter of $\mathbb{P}$ namely $\delta_{\mathbb{P}}:=\sup_{(x_1,x_2)\in\mathbb{P}}\|x_1-x_2\|$. For a compact set $\mathbb{X}$, the notation $\varrho(\mathbb{X})$ denotes the maximum norm of elements in $\mathbb{X}$, namely $\varrho(\mathbb{X}):=\sup_{x\in\mathbb{X}}\|x\|$. p_u denotes the unconstrained minimum of f_0. It is assumed that the functions involved in (5) meets the following assumption:

Assumption 2.1: There exist strictly positive reals $L_0, L_\psi, \mu_0, \beta$ such that $f_0\in\mathcal{S}^1_{\mu_0,L_0}$, $\psi\in\mathcal{F}^1_{L_\psi}$. Moreover, f_0 is proper, nonnegative, and ψ is convex and such that the following inequality:

$$\psi(p)\ge \beta\times[d(p,\mathcal{A}_{\psi=0})]^2 \quad (9)$$

holds for all p. ⊙

Note that this assumption implies that p^* is a unique stationary point for f and that $d(p):=\|p-p^*\|$. Let p_a be any admissible point, that is $\psi(p_a)=0$. Let $D_0>0$ be such that

$$D_0\ge \sup_{f_0(p)\le f_0(p_a)}\|f_0'(p)\|\ge 0. \quad (10)$$

Note that Assumption 2.1 implies that $f=f_0+\rho\psi$ belongs to $\mathcal{F}^1_L$ with $L=L_0+\rho L_\psi$. The following function is used in the statement of the certification result:

$$Z(\epsilon):=\frac{D_0}{L_0}\left[\left(1+\frac{2L_0}{D_0^2}\epsilon\right)^{\frac{1}{2}}-1\right]. \quad (11)$$

B. Certifiable FG-Algorithm to Solve (4)

For the sake of completeness, **Algorithm 1** recalls the definition of the FG iteration when used to seek a minimum of f defined by (5).

Algorithm 1 $[p_{i+1}, q_{i+1}, \alpha_{i+1}] = F^{(1)}(p_i, q_i, \alpha_i)$

1: $p_{i+1}\leftarrow q_i - f'(q_i)/L$
2: Compute α_{i+1} s.t $\alpha_{i+1}^2=(1-\alpha_{i+1})\alpha_i^2+\mu_0\alpha_{i+1}/L$
3: $\beta_i\leftarrow(\alpha_i(1-\alpha_i))/(\alpha_i^2+\alpha_{i+1})$
4: $q_{i+1}\leftarrow p_{i+1}+\beta_i(p_{i+1}-p_i)$

Note that p, q, and α are internal states whose initialization is explicitly given in **Algorithm 2** hereafter. This algorithm is defined for a given initial guess p_0 and a given pair $\bar{\epsilon}:=(\epsilon_0,\epsilon_\psi)$ of targeted precision. Moreover, the number of iterations $N_{\max}$ invoked in step 6 are defined by

$$\bar{N}(c,\gamma):=\max\left\{0,\min\left\{\frac{\log(\gamma)}{\log(1-c)},\frac{1}{c}\left(\sqrt{\frac{1}{\gamma}}-1\right)\right\}\right\}. \quad (12)$$

Note also that before the fast gradient iterations are fired, the algorithm sets (in step 2) the value of the weighting parameter ρ to be used in the definition of $f(p)$ involved in (5). Regarding **Algorithm 2**, the following result holds:

Algorithm 2 $\hat{p}^* = A(p_0, \bar{\varepsilon} := (\varepsilon_0, \varepsilon_\psi))$

1: $\alpha_0=(\mu_0/L)^{1/2}$, $q_0:=p_0$, again = true.
2: $\rho=\max\{(2L_\psi\kappa_0^2/\varepsilon_\psi^2),(\mu_0\varepsilon_\psi^2/4L_\psi),L_0/\beta\}$ where $\kappa_0:=(2L_0/\beta)\sqrt{(2/\mu_0)\psi(p_u)}$
3: $\eta=\min\{(\mu_0/2)Z^2(\varepsilon_0/2),(\mu_0\varepsilon_\psi^2/4L_\psi)\}$
4: $c=(\mu_0/L)^{1/2}$
5: $\gamma=\eta\mu_0/[(L+\mu_0)f_0(p_0)]$
6: $N_{\max}=\bar{N}(c,\gamma)$
7: $g_{\min}=\mu_0(2\eta/L)^{1/2}$
8: **while** (again) **do**
9: $[p_{i+1},q_{i+1},\alpha_{i+1}]=F^{(1)}(p_i,q_i,\alpha_i)$
10: **if** $[(i\ge N_{\max})$ or $(g(p_i)\le g_{\min})]$ **then** again = false
11: **else** $i=i+1$
12: **end if**
13: **end while**
14: $\hat{p}^*=p_i$

Proposition 2.1: Let be given a precision pair $\bar{\varepsilon}:=(\varepsilon_0,\varepsilon_\psi)$ and some initial guess p_0. Define $c=(\mu_0/L)^{1/2}$ and $\gamma=\eta\mu_0/[(L+\mu_0)f_0(p_0)]$. **Algorithm 2** involves at most $\bar{N}(c,\gamma)=: N(p_0,\epsilon_0,\epsilon_\psi)$ [see (12)] unconstrained fast gradient elementary iterations (Algorithm 1) before it delivers an estimate $\hat{p}^*$ that is an $\bar{\varepsilon}$-suboptimal solution of the original constrained problem (4). ⊙

Due to space limitation, the proof can be consulted in [2] where a complete instantiation of the result is given for quadratic problems together with an evaluation of the ratio between the certified number of iterations and the effectively needed number of iterations for a randomly generated set of QP problems. Note that **Algorithm 2** can also be stopped by the condition on the gradient [see step 10:] in order to avoid useless iterations.

III. Certifiable Real-Time Interruptible MPC

A. Working Assumptions

Note that the development of Section II applies to a convex optimization problem of the form (4) which is state-independent. When certification bounds are to be invoked for state-dependent online problems of the form (3), for instance in MPC context, the number of iterations takes the form $N(p,\epsilon_0,\epsilon_\psi,x)$ since some of the characteristic quantities (such as L_0, L_ψ, β) that determined the certification bound are now state-dependent. It is shown in [2] that a (p, x)-independent certification bounds can be derived for the FG-based algorithm proposed in Section II for any given compact subset $\mathbb{C} \subset \mathbb{P} \times \mathbb{X}$ to which belongs (p, x) involved in $N(p, \epsilon_0, \epsilon_\psi, x)$.

As far as this section is concerned and in order to keep the contribution of the present section independent of the specific use of the certification bound proposed in the previous section, the availability of such a certification bound that depends only on the precision pair $(\epsilon_0, \epsilon_\psi)$ and the compact set $\mathbb{C}$ is viewed here as a necessary assumption, namely:

Assumption 3.1 (Availability of Certifiable Algorithm): The algorithm A used to solve (3) is such that, given a compact subset $\mathbb{C} \subset \mathbb{P} \times \mathbb{X}$ and a targeted precision $\bar{\epsilon} := (\epsilon_0, \epsilon_\psi)$, an $\bar{\epsilon}$-suboptimal solution to (3) can be obtained in less than $N_\mathbb{C}(\epsilon_0, \epsilon_\psi)$ for all $(p, x) \in \mathbb{C}$. Moreover, the hardware performs a single iteration of A in τ_c time units. $\odot$

As the state x is supposed to gather the physical state, the set-point current value and the current estimation of uncertainty, the prediction used in the MPC computation is inherently uncertain. This is stated in the following assumption:

Assumption 3.2: For each compact set $\mathbb{C}$ to which belongs the pair $(p(t), x(t))$, the prediction $\hat{x}(t+\tau)$ of the future state starting from $x(t)$ and under the control profile $\mathcal{U}(\cdot, p(t))$ can be affected by an error satisfying the following inequality:

$$\|\hat{x}(t+\tau) - x(t+\tau)\| \leq E^0_\mathbb{C} + E^1_\mathbb{C} \times \tau. \tag{13}$$

Note that the prediction of $\hat{x}(t+\tau)$ is a part of the available computation time and some fast dedicated schemes such as the one proposed in [14] can be appropriately used if necessary. The following assumption is needed regarding the cost function f_0 and the constraints-induced map ψ:

Assumption 3.3 (Properties of f_0 and ψ): For any positive real $\phi > 0$, there is a compact set $\mathbb{C}_\phi$ such that

$$\{f_0(p, x) \leq \phi\} \Rightarrow \{(p, x) \in \mathbb{C}_\phi\}. \tag{14}$$

Moreover, for any compact set $\mathbb{C}$, there are positive real Lypschitz continuity coefficient $K^0_\mathbb{C}, K^\psi_\mathbb{C} > 0$ s.t.

$$\|f_0(p, x_1) - f_0(p, x_2)\| \leq K^0_\mathbb{C} \cdot \|x_1 - x_2\| \tag{15}$$

$$\|\psi(p, x_1) - \psi(p, x_2)\| \leq K^\psi_\mathbb{C} \cdot \|x_1 - x_2\| \tag{16}$$

for all $(p, x_1), (p, x_2) \in \mathbb{C}$. $\odot$

A typical formulation of the cost function $f_0(p, x_0)$ in MPC is given by:

$$f_0(p, x_0) := \Omega\left(\bar{x}(T, p, x_0)\right) + \int_0^T \ell\left(\bar{x}(s, p, x_0), p, s\right) ds$$

$$=: \Omega\left(\bar{x}(T, p, x_0)\right) + \int_0^T \bar{\ell}(s, p, x_0) ds \tag{17}$$

where $\bar{x}(s, p, x_0)$ is the predicted state value at instant s starting from x_0 at instant 0 and under the control sequence defined by p while T is the prediction horizon.

Regarding the formulation of the MPC, the following (commonly satisfied) assumption is needed in the sequel:

Assumption 3.4 (Properties of Ideal MPC): The MPC formulation is based on a cost function of the form (17) with the necessary constraints that make the following Lyapunov inequality satisfied:

$$f_0\left(p^{\text{opt}}(t+\tau), \hat{x}(t+\tau)\right) - f_0\left(p^{\text{opt}}(t), x(t)\right)$$

$$\leq -\Delta\left(\tau, x(t)\right) := -\int_0^\tau \bar{\ell}\left(s, p^{\text{opt}}(t), x(t)\right) ds \tag{18}$$

where $p^{\text{opt}}(t)$ is an optimum of the problem defined for the state $x(t)$ while $p^{\text{opt}}(t+\tau)$ is an optimum of the problem defined by the predicted future state $\hat{x}(t+\tau)$ starting from $x(t)$ under the control profile $\mathcal{U}(\cdot, p^{\text{opt}}(t))$ that is applied on the interval $[t, t+\tau]$. $\odot$

Note that $p^{\text{opt}}(t)$ does not appear as an argument of Δ since $p^{\text{opt}}(t)$ is assumed to be uniquely determined by $x(t)$.

Remark 3.1: The inequality (18) is satisfied only for the ideal predicted future state $\hat{x}(t+\tau)$ since otherwise the bad knowledge of uncertainties and/or the set-point changes may invalidate (18) should the true value $x(t+\tau)$ be used.

Remark 3.2: Note that inequality (18) is commonly satisfied in the standard provably stable MPC formulations. Moreover, the r.h.s $\Delta(\tau, x(t))$ is generally exhibited through the corresponding stability proof (see [6]).

Regarding the penalty function ℓ, the following assumption is used:

Assumption 3.5 (Fig. 1): For any compact set $\mathbb{C}$, there is a positive real $D_\mathbb{C} > 0$ and a positive function $q(\cdot)$ such that:

$$\bar{\ell}(s, p, x) \geq \max\{0, q(x) - D_\mathbb{C} s\} \tag{19}$$

for all $(p, x) \in \mathbb{C}$. $\odot$

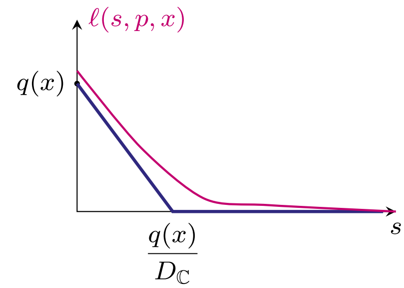

Fig. 1. Illustration of Assumption 3.5.

Note that condition (19) simply states that with bounded control, there is a limitation on the speed with which the state can be steered to the desired region. In this respect, $q(x)$ is simply a state-dependent term in ℓ that expresses how far does x lie from the desired region.

B. Main Results

Assume that a scheme is based on the iterative on-line definition of a sequence of updating instants and a sequence of precision parameters denoted by

$$t_{k+1} = t_k + \tau_k; \quad \left\{\varepsilon_0^{(k)}, \varepsilon_\psi^{(k)}\right\}_{k=0}^\infty \tag{20}$$

which are linked through the definition of the updating periods τ_k according to

$$\tau_k := \tau_c \times N_\mathbb{C}\left(\varepsilon_0^{(k+1)}, \varepsilon_\psi^{(k+1)}\right) \tag{21}$$

where $\mathbb{C}$ is some compact subset of $\mathbb{R}^{n_p} \times \mathbb{R}^n$ and τ_c is the computation time needed for a single iteration (see Assumption 3.1).

More precisely, given the current state $x(t_k)$ and a control $\mathcal{U}(\cdot, \hat{p}^*(t_k))$ that is applied during the sampling period $[t_k, t_{k+1}]$, Algorithm 2 is used to compute the control parameter $\hat{p}^*(t_{k+1})$ (that is to be applied during the next sampling period) with the hot start

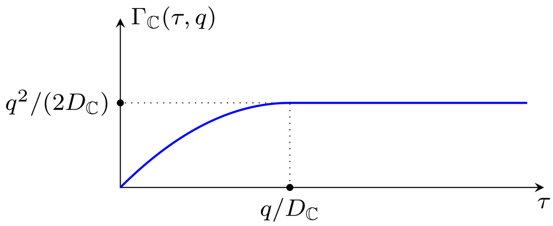

Fig. 2. Evolution of $\Gamma_{\mathbb{C}}(\tau, q)$ involved in Lemma 3.2.

$[\hat{p}^*(t_k)]^{+\tau_k}$ and the precision parameters $(\varepsilon_0^{(k+1)}, \varepsilon_\psi^{(k+1)})$. Note that by the very definition (21) of τ_k, the value of the control parameter $\hat{p}^*(t_{k+1})$ that is obtained by Algorithm 2 before t_{k+1} necessarily meets the precision requirements, namely

$$f_0\left(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})\right) - f_0\left(p^{\text{opt}}(t_{k+1}), \hat{x}(t_{k+1})\right) \le \varepsilon_0^{(k+1)}$$
$$c_i\left(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})\right) \le 0 \qquad i \in I_h$$
$$c_i\left(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})\right) \le \varepsilon_\psi^{(k+1)} \qquad i \in I_s. \tag{22}$$

Using the first inequality, one can prove the following result:

Lemma 3.1: **If** the following conditions hold

1) τ_k is defined by (21) for some compact set $\mathbb{C} := \mathbb{P} \times \mathbb{X}$
2) For all k, $[\hat{p}^*(t_k)]^{+\tau_k} \in \mathbb{P}$
3) For all k, $x(t_k) \in \mathbb{X}$
4) Assumptions 3.2, 3.3, and 3.4 are satisfied

then the following inequality holds for all k:

$$f_0\left(\hat{p}^*(t_{k+1}), x(t_{k+1})\right) - f_0\left(\hat{p}^*(t_k), x(t_k)\right)$$
$$\le \varepsilon_0^{(k)} + K_{\mathbb{C}}^0\left(E_{\mathbb{C}}^0 + E_{\mathbb{C}}^1 \tau_k\right) + \varepsilon_0^{(k+1)} - \Delta\left(\tau_k, x(t_k)\right). \tag{23}$$

Proof: See Appendix A.

Note that the term $f_0(\hat{p}^*(t_k), x(t_k))$ represents the value of the cost function at the effectively *visited* pairs $(\hat{p}(t_k), x(t_k))$. Therefore, the difference expressed in the l.h.s of (23) is relevant for the stability assessment of the resulted truncated MPC implementation. On the other hand, using the definition (21) of τ_k, the r.h.s of (23) can be viewed as a function of the precision pair $(\varepsilon_0^{(k+1)}, \varepsilon_\psi^{(k+1)})$. The stability issue is therefore dependent on the possibility to define these precision parameters in such a way that the r.h.s of (23) is negative. This is the aim of the following development.

Remark 3.3: In the sequel, only ε_0 is made state-dependent as ϵ_ψ is supposed to be sufficiently small so that the recursive feasibility (generally associated to the satisfaction of some constraints) holds for the interrupted-MPC trajectories. Note that for this to be rigorously true, these original constraints have to be ϵ_ψ-tightened which is implicitly assumed in the reminder of the technical note.

Since the only negative term in the RHS of (23) is $-\Delta(\tau_k, x(t_k))$, we need a lower bound on $\Delta(\tau_k, x(t_k))$. The following straightforward lemma gives such a lower bound:

Lemma 3.2: **If** the following conditions hold:

1) $(\hat{p}^*(t_k), x(t_k)) \in \mathbb{C}$
2) Assumption 3.5 is satisfied

then a computable lower bound of the quantity $\Delta(\tau, x(t_k))$ can be obtained by:

$$\Delta\left(\tau, x(t_k)\right) \ge \Gamma_{\mathbb{C}}\left(\tau, q(x(t_k))\right) \tag{24}$$

where $\Gamma_{\mathbb{C}}(\tau, q)$ is given by (see Fig. 2)

$$\Gamma_{\mathbb{C}}(\tau, q) := \begin{cases} q\tau - \frac{1}{2}D_{\mathbb{C}}\tau^2 & \text{if } \tau \le q/D_{\mathbb{C}} \\ \frac{q^2}{2D_{\mathbb{C}}} & \text{otherwise.} \end{cases} \tag{25}$$

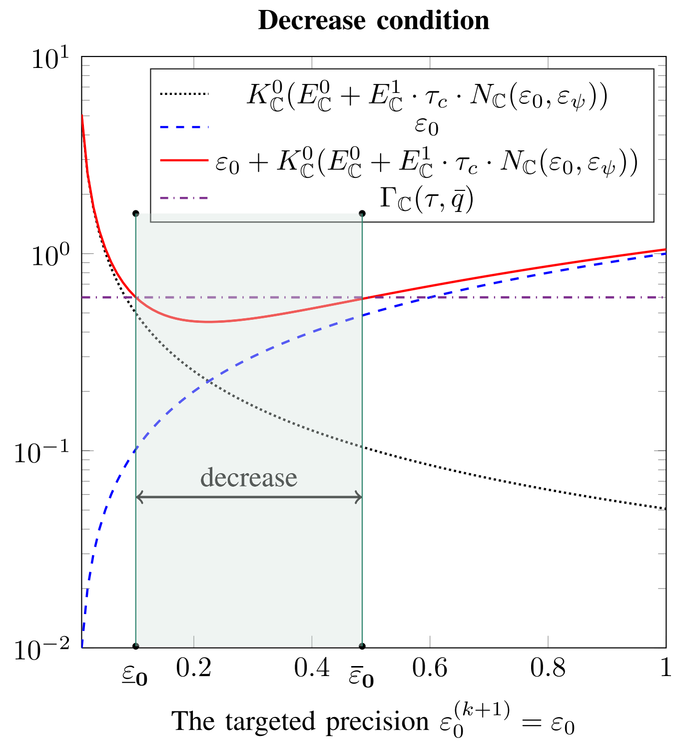

Fig. 3. Typical evolution of the quantities involved in the r.h.s of (27) invoked in corollary 1. The decrease of the cost function is possible if there is a tagreted future precision ε_0 for which the red-solid curve lies below the dash-dotted curve. Note that the past value of ε_0 is taken equal to 0 for simplicity.

Proof: See Appendix B.

Using the definition (21) of τ_k and the r.h.s of (24) in (23) the following computable function can be defined:

$$R_{\tau_c}(\varepsilon_0, \varepsilon_\psi, \bar{q}) := K_{\mathbb{C}}^0\left(E_{\mathbb{C}}^0 + \tau_c E_{\mathbb{C}}^1 N_{\mathbb{C}}(\varepsilon_0, \varepsilon_\psi)\right)$$
$$+ \varepsilon_0 - \Gamma_{\mathbb{C}}\left(\tau_c \cdot N_{\mathbb{C}}(\varepsilon_0, \varepsilon_\psi), \bar{q}\right) \tag{26}$$

so that the following corollary of Lemma 3.1 can be stated:

Corollary 1: **If** the following conditions hold

1) The requirements of Lemma 3.1 are satisfied
2) Assumption 3.5 holds
3) $q(x(t_k)) \ge \bar{q}$

then the following inequality holds:

$$f_0\left(\hat{p}^*(t_{k+1}), x(t_{k+1})\right) - f_0\left(\hat{p}^*(t_k), x(t_k)\right)$$
$$\le \varepsilon_0^{(k)} + R_{\tau_c}\left(\varepsilon_0^{(k+1)}, \varepsilon_\psi^{(k+1)}, \bar{q}\right) \tag{27}$$

where $R_{\tau_c}(\cdot)$ is defined by (26).

Fig. 3 presents a typical situation showing that even for a past achieved precision $\varepsilon_0^{(k)} = 0$, a given computational power leading to the computation time τ_c and a given precision ε_ψ on the soft constraints satisfaction, either there is no $\varepsilon_0^{(k+1)}$ making the r.h.s of (26) invoked in corollary 1 negative or there is an interval of successful values of $\varepsilon_0^{(k+1)}$ which does not contain 0 and which depends on the current value of $q(x(t_k)) = \bar{q}$.

Note that Corollary 1 involves quantities that depend on some compact set to which belong all the pair $([\hat{p}^*(t_k)]^{+\tau_k}, \hat{x}(t_{k+1})$. Using Assumption 3.3, it is possible to prove that such compact set is linked to a set of initial conditions for which a certified convergence result can be derived for the resulting real-time MPC. This is stated in the following proposition which is the main contribution of the technical note:

Proposition 3.1: Consider a positive real $\phi_0 > 0$ and the corresponding compact subset $\mathbb{C}_{\phi_0} \subset \mathbb{R}^{n_p} \times \mathbb{R}^n$ defined according to Assumption 3.3. Let be given a precision $\varepsilon_\psi > 0$ on the soft constraints satisfaction.

If the following conditions hold with $\mathbb{C} = \mathbb{C}_{\phi_0}$:

1) Assumptions 3.1–3.5 are satisfied
2) $\exists \bar{q}_{\min} > 0$ and $\gamma_c > 0$ such that the inequality

$$R_{\tau_c}(\varepsilon_0, \varepsilon_\psi, \bar{q}) \leq -\left[\frac{\gamma_c \bar{q}^2_{\min}}{3D_{\mathbb{C}_{\phi_0}}}\right] \tag{28}$$

admits a solution $\varepsilon_0^{\text{sol}}(\bar{q}) \in [0, \gamma_c \bar{q}^2_{\min}/(2D_{\mathbb{C}_{\phi_0}})]$ for all $\bar{q} \geq \bar{q}_{\min}$

then the truncated MPC design based on the adaptive sampling period defined by

$$\tau_k := \tau_c \times N_{\mathbb{C}}\left(\varepsilon_0^{\text{sol}}\left(q\left(x(t_k)\right)\right), \varepsilon_\psi\right) \tag{29}$$

steers the system to the set

$$\mathbb{X}_{\min} := \{x \in \mathbb{R}^n \quad | \quad q(x) \leq \bar{q}_{\min}\} \tag{30}$$

provided that the initial condition satisfies

$$f_0\left(\hat{p}^*(t_0), x(t_0)\right) \leq \phi_0; \quad \varepsilon_0^{(0)} \leq \frac{\gamma_c \bar{q}^2_{\min}}{6D_{\mathbb{C}_{\phi_0}}}. \tag{31}$$

Moreover, if the hard constraints depend only on p, then along the closed-loop trajectory, one has

$$\max_{i \in I_h, k \geq 0} \left[c_i\left(\hat{p}^*(t_k), x(t_k)\right)\right] \leq 0$$
$$\max_{i \in I_s, k \geq 0} \left[c_i\left(\hat{p}^*(t_k), x(t_k)\right)\right] \leq \varepsilon_\psi + K^\psi_{\mathbb{C}_{\phi_0}} \cdot \left(E^0_{\mathbb{C}_{\phi_0}} + E^1_{\mathbb{C}_{\phi_0}} \tau_k\right). \tag{32}$$

Proof: See Appendix C.

In [2], a complete computation of the state-dependent CUP map $\varepsilon_0^{\text{sol}}(q(x))$ is conducted for a linear MPC example. This could not be reproduced in the present technical note due to space limitations.

IV. Conclusion and Future Work

In this technical note, a deep understanding of the interaction between the different parameters involved in real-time implementation of interruptible MPC schemes is provided. This leads to state-dependent control updating period that can be designed for a specific algorithm with its associated certification bound, a specific hardware efficiency, a specific MPC ideal provably stable (in the ideal case) formulation and a specific bounding description of the uncertainties.

Undergoing work concerns a step forward towards the optimality of the CUP choice. Indeed, Proposition 3.1 only states the existence of an interval to which an appropriate CUP can be found such that the asymptotic attractive set can be guaranteed. The implementation proposed in [2] gives only a possible choice among many others. A more near-to-optimal choice of the map $\varepsilon_0^{\text{sol}}(q(x))$ remains to be precisely derived. This can be for instance done, following the cost function defined in [1] but using the analytic expression derived in the present technical note while iterative blind gradient-based iteration were used in [1].

Appendix

A. Proof of Lemma 3.1

Using Assumption 3.2 and 3.3, it comes that

$$f_0\left(\hat{p}^*(t_{k+1}), x(t_{k+1})\right) \leq f_0\left(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})\right) + K^0_{\mathbb{C}} \times \left[E^0_{\mathbb{C}} + E^1_{\mathbb{C}} \times \tau_k\right]. \tag{33}$$

Now by definition of τ_k, the solution $\hat{p}^*(t_{k+1})$ satisfies

$$f_0\left(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})\right) \leq f_0\left(p^{\text{opt}}(t_{k+1}), \hat{x}(t_{k+1})\right) + \varepsilon_0^{(k+1)}$$

which together with Assumption 3.4 gives

$$\begin{aligned} f_0(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})) &\leq f_0\left(p^{\text{opt}}(t_k), x(t_k)\right) + \varepsilon_0^{(k+1)} - \Delta(\tau_k, x(t_k)) \\ &\leq f_0\left(\hat{p}^*(t_k), x(t_k)\right) + \varepsilon_0^{(k)} + \varepsilon_0^{(k+1)} \\ &\quad - \Delta\left(\tau_k, x(t_k)\right). \end{aligned} \tag{34}$$

Using the last inequality in (33) gives the result. □

B. Proof of Lemma 3.2

By definition of (18) of Δ and using (19) of Assumption (3.5), it comes that

$$\begin{aligned} \Delta(\tau, x) &\geq \int_0^\tau \max\{0, q(x) - D_{\mathbb{C}} s\}\, ds \\ &= \int_0^{\min\{\tau, q(x)/D_{\mathbb{C}}\}} (q(x) - D_{\mathbb{C}} s)\, ds \\ &= \left[q(x)\tau - \frac{1}{2} D_{\mathbb{C}} \tau^2\right]_0^{\min\{\tau, q(x)/D_{\mathbb{C}}} \end{aligned}$$

which can be expressed using $\Gamma_{\mathbb{C}}(\tau, q)$ given by (25). □

C. Proof of Proposition 3.1

The first inequality in (31) together with Assumption 3.3 impliy that Corollary 1 applies with $k = 0, \mathbb{C} = \mathbb{C}_{\phi_0}$ and $\bar{q} := q(x(t_k))$, therefore one has

$$f_0\left(\hat{p}^*(t_1), x(t_1)\right) - f_0\left(\hat{p}^*(t_0), x(t_0)\right) \leq \varepsilon_0^{(0)} + R_{\tau_c}\left(\varepsilon_0^{(1)}, \varepsilon_\psi, q\left(x(t_0)\right)\right) \tag{35}$$

and since $\varepsilon_0^{(1)} = \varepsilon_0^{\text{sol}}(x(t_0))$, if $q(x(t_0)) > \bar{q}_{\min}$ the inequality (28) gives

$$R_{\tau_c}\left(\varepsilon_0^{(1)}, \varepsilon_\psi, q\left(x(t_0)\right)\right) \leq -\frac{\gamma_c \bar{q}^2_{\min}}{3D_{\mathbb{C}_{\phi_0}}} \tag{36}$$

and thanks to the second inequality in (31), the inequality (36) gives

$$\varepsilon_0^{(0)} + R_{\tau_c}\left(\varepsilon_0^{(1)}, \varepsilon_\psi, q\left(x(t_0)\right)\right) \leq -\frac{\gamma_c \bar{q}^2_{\min}}{6D_{\mathbb{C}_{\phi_0}}}. \tag{37}$$

This, together with (35), implies that $f_0(\hat{p}^*(t_1), x(t_1))$ decreases meaning that the new pair is still in $\mathbb{C}_{\phi_0}$ and since $\varepsilon_0^{(1)}$ satisfies by assumption the second inequality in (31), the argumentation can be repeated to derive the properties of the next pair $(\hat{p}^*(t_2), x(t_2))$ meaning that the following inequality:

$$f_0\left(\hat{p}^*(t_{k+1}), x(t_{k+1})\right) - f_0\left(\hat{p}^*(t_k), x(t_k)\right) \leq -\frac{\gamma_c \bar{q}^2_{\min}}{6D_{\mathbb{C}_{\phi_0}}}$$

is satisified as far as $q(x(t_k))$ remains grater than $\bar{q}_{\min}$. This clearly implies that $x(t_k)$ converges to the limit set $\mathbb{X}_{\min}$ defined by (30).

Regarding the constraints, note that the hard constraints are necessarily satisfied since they depend only on p by assumption and that $\hat{p}^*(t_{k+1})$ satisfies by construction the hard constraints while allowing only for a violation of the soft constraints by an amount which is lower than ε_ψ, therefore, one has

$$c_i\left(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})\right) \leq \varepsilon_\psi \quad \forall\, i \in I_s \tag{38}$$

which obviously gives (32) by Assumptions 3.2 and 3.5. □

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