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Stability of a Truncated Infinite Constrained Receding Horizon Scheme: the General Discrete Nonlinear Case*

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Key Words—Receding horizon; stability; truncation; constrained optimal control; prediction horizon; control horizon.

Abstract—Some results concerning the constrained receding horizon formulation with infinite and truncated prediction horizon are presented. In this formulation a distinction is made between prediction and control horizons. The main result is a generalization of a known fact for linear systems, namely that, under certain technical requirements, the asymptotic stability of the above formulation holds for a sufficiently large finite prediction horizon. Existence results are also provided under stabilizability assumptions.

1. *Introduction*

Receding horizon optimal control strategy has been largely studied for linear systems. It presents an alternative approach to geometric methods, which are very sensitive to the structural properties of the system. Furthermore, it presents a conceptually attractive way to handle state and input constraints.

In this strategy one determines a control sequence that minimizes some open-loop performance index on a time interval of length equal to the prediction horizon. The first input of the above sequence is implemented so as to move to the next state. The whole computation is then repeated at the new state (which is supposed to be measurable) in order to obtain closed-loop behavior. The length of the computed input sequence is called the control horizon; it represents the dimension of the optimization problem.

In the case of linear systems an enormous amount of work exists, discussion of which is beyond the scope of this paper. Garcia *et al.* (1989) gave a survey of what was done in the preceding decade. For more recent work see Rawlings and Muske (1993), Muske and Rawlings (1992) and references therein.

Bitmead *et al.* (1990) the authors underlined the poor stability properties of finite-prediction-horizon schemes. Two methods have been proposed to achieve stability. The first is to add final constraint on the state vector. The second is to use infinite-prediction-horizon formulation. Mayne and Michalska (1990) and Michalska and Mayne (1991) proposed a formulation with finite prediction horizon and constrained final state for continuous nonlinear systems, with stability being obtained under certain controllability and regularity assumptions. A roughly similar formulation was used by Alamir and Bornard (1994a) for discrete-time nonlinear systems. More recently, in order to avoid the numerical difficulties when performing optimization with equality constraints, derived formulations based on final inequality constraints have been proposed by Michalska and Mayne (1993) for continuous systems and Alamir and Bornard (1994b) for discrete systems.

The use of an infinite-prediction-horizon scheme has been proposed by Keerthi and Gilbert (1987, 1988) for discrete-time nonlinear systems. In this work no distinction is made between prediction and control horizons. The stability of the infinite-horizon scheme is proved under a controllability assumption; then the existence of a finite horizon that conserves the stability is proved. It seems natural to expect such a finite stabilizing prediction horizon to be quite large; that is why the use of *a priori* identical prediction and control horizons should be reexamined.

In this paper we consider general discrete-time nonlinear systems. The stability of an infinite-prediction-horizon and finite-control-horizon scheme is first discussed. It turns out in particular that stability holds under the stabilizability assumption above, and that there is no need to take a control horizon as long as the prediction horizon. Indeed, the control horizon length achieves the feasibility of the strategy while the prediction horizon achieves the global stability requirements. Then we prove that, under certain regularity conditions that seem to be of a technical nature, the stability results are open with respect to the prediction horizon at infinity. In other words, one can always find a finite prediction horizon that makes the strategy globally asymptotically stabilizing.

It is worth noting that Gauthier and Bornard (1983) were the first to establish stability of the receding horizon controller using a finite control horizon and infinite prediction horizon for linear quadratic unconstrained problem. Rawlings and Muske (1993) reconsidered the problem in a slightly different formulation.

The existence of a finite prediction horizon such that the corresponding receding horizon controller is stabilizing has already appeared in the literature dealing with linear systems (Garcia *et al.*, 1989). This property, which is quite intuitive, has never been reported after—to the best of our knowledge—in the literature on general nonlinear systems. Here we give sufficient conditions under which this property holds.

The paper is organized as follows. Some definitions and notations are given in Section 1. In Section 2 the feasibility and the stability of an infinite-prediction–finite-control horizon strategy is then discussed. The truncation of the prediction horizon is studied in Section 4.

It is worth noting that the whole development in this paper also holds for an output tracking scheme with reference model. In order to simplify the notation and underline the main ideas, a simple regulation problem is presented. The complete development in a tracking configuration can be found in Alamir and Bornard (1993).

2. *Definitions and notation*

We consider a discrete-time system described by

$$x(k+1) = f(x(k), u(k)), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control input. We

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assume that f is continuously differentiable, with $f(0,0)=0$. The regions of admissible states and controls are given by the compact sets $\Omega_x \subset \mathbb{R}^n$, and $\Omega_u \subset \mathbb{R}^m$ respectively. The admissible regions are supposed to be such that $\Omega_x \times \Omega_u$ contains a neighborhood of the origin.

Let $x(k; x_0; \tilde{u})$ denote the solution of (1) at instant k, with the initial condition $x(0)=x_0$ and due to the sequence $\tilde{u} \in (\mathbb{R}^m)^{\mathbb{N}}$. The subset of admissible control sequences is therefore given by

$$U_N^\infty(x) = \{\tilde{u} \in \Omega_u^{\mathbb{N}} \mid \tilde{u}_{N+i} = \tilde{u}_N \ \forall i \geq 0 \text{ and } x(k; x; \tilde{u}) \in \Omega_x \ \forall k \geq 0\}, \tag{2}$$

which is clearly the set of all sequences of admissible controls that becomes constant after N steps and corresponding to admissible trajectories. Receding horizon control at state x is obtained by solving the finite-dimensional constrained problem $P_N^M(x)$ defined by

$$P_N^M(x) = \min_{\tilde{u} \in U_{ad}^M(x)} J_N^M(x, \tilde{u}) \tag{3}$$

where $J_N^M(x, \tilde{u})$ is the performance index defined over the horizon $M \in \mathbb{N} \cup \{\infty\}$, $U_{ad}^M(x)$ is the set of admissible control sequences rendering the performance index finite. That is,

$$U_{ad}^M(x) = \{\tilde{u} \in U_N^\infty(x) \mid J_N^M(x, \tilde{u}) < \infty\}. \tag{4}$$

We shall refer to M and N as the prediction horizon and control horizon respectively. The solution of (3)—if it exists—will be denoted by $\hat{u}^M(x)$, the corresponding optimal open-loop trajectory by $\hat{x}^M(x)$ and the optimal value of the performance index by $\hat{J}_N^M(x)$. When the solution $\hat{u}^M(x)$ exists, the receding horizon control law $h^M(x)$ is given by

$$h^M(x) = \hat{u}_0^M(x), \tag{5}$$

where $\hat{u}_0^M(x)$ is the first control vector in the optimal sequence $\hat{u}^M(x)$. Note that the subscript N is omitted in the definition of the optimal solution; that is because N will be fixed throughout the paper. This enables to use the subscript to indicate the rank of vectors in the sequence $\hat{u}^M(x)$.

We shall use the performance index given by

$$J_N^M(x, \tilde{u}) = \sum_{k=0}^{M} \|x(k; x; \tilde{u})\|_Q^2 + \|\tilde{u}_k\|_R^2 \tag{6}$$

where Q and R are strictly positive-definite matrices.

The aim of this paper is, first, to find sufficient conditions under which the problem (3) has a solution, and such that the closed-loop system $x(k+1) = f(x(k), h^\infty(x(k)))$ is globally asymptotically stable. Secondly it is to find conditions that make it possible to truncate the prediction horizon at a finite value of M without altering the stability, that is, to find sufficient conditions under which there exists a finite M such that $x(k+1) = f(x(k), h^M(x(k)))$ is globally asymptotically stable.

3. *Global stability in the infinite case* ($M = \infty$)

We shall make the following assumption.

Assumption 3.1. $\exists N \in \mathbb{N}$ s.t, for all $x_0 \in \Omega_x$, $U_{ad}^\infty(x_0) \neq \emptyset$.

This means that there exist admissible control sequences that rapidly steer the system to the origin for all initial state. In what follows, N denotes a fixed integer satisfying Assumption 3.1. We have the following proposition.

Proposition 3.1. Under Assumption 3.1, the problem $P_N^\infty(x)$ has a solution for all $x \in \Omega_x$.

Sketch of the proof. (See Alamir and Bornard (1993).) Let $\tilde{u}^* \in U_{ad}^\infty(x)$. It is clear that any candidate solution must belong to

$$\mathcal{U}_{ad} = \{\tilde{u} \in U_{ad}^\infty(x) \mid J_N^\infty(x, \tilde{u}) \leq J_N^\infty(x, \tilde{u}^*)\}.$$

Therefore the problem $P_N^\infty(x)$ can be written equivalently as

$$P_N^\infty(x): \min_{\tilde{u} \in \mathcal{U}_{ad}} J_N^\infty(x, \tilde{u}).$$

However, $\mathcal{U}_{ad}$ is clearly a compact set, and this, with the continuity of $J_N^M(x, .)$ for all $m \in \mathbb{N}$, gives the result.

The following is a classical result in the receding horizon literature (see e.g. Alamir and Bornard, 1993):

Proposition 3.2. Under Assumption 3.1, the functions

$$V(x) = \hat{J}_N^\infty(x),$$
$$W(x) = \|x\|_Q^2 + \|\hat{u}_0^\infty(x)\|_R^2$$

are well defined and such that

$$V(\hat{x}_1^\infty(x)) - V(x) \leq -W(x) \leq 0.$$

Furthermore, for all initial states, the trajectory of the closed-loop system defined by the feedback law $h^\infty(x)$ is bounded.

This last proposition proves that V is a Lyapunov function for the closed-loop trajectory, and leads to the following theorem (see Alamir and Bornard, 1993).

Theorem 3.1. Under Assumption 3.1, the strategy defined by (5) with $M = \infty$ leads to a closed-loop system that is globally asymptotically stable.

4. *Truncation of the prediction horizon*

In this section we prove that, under certain assumptions, there is no need to keep an infinite prediction horizon to make the strategy stabilizing. The proof includes two steps. The first relies upon the fact that, however small a neighborhood of zero is taken, there is always a finite horizon length M that corresponds to a strategy that forces the state of the closed-loop system to reach that neighborhood.

The second step of the proof consists of local considerations. Indeed, such considerations enable us to prove that *locally* the stability of a receding infinite-prediction-horizon control strategy is open with respect to the horizon length at infinity. This, with the result of the first step, enables us to prove the result.

4.1. *Definitions and preliminary results.*

Definition 4.1. Let $\phi_N: \mathbb{R}^n \times \mathbb{R}^{Nm} \to \mathbb{R}^n$ be given by

$$\phi_N(x, \tilde{u}) = x(N; x; \tilde{u}). \tag{7}$$

We suppose that the following regularity assumption holds.

Assumption 4.1. For all $x \in \Omega_x$ there exists $r > 0$ s.t, for all $M \geq N$,

$$\text{Rank}\left[\frac{\partial \phi_N}{\partial \tilde{u}}(x, \hat{u}^M(x))\right] = n, \tag{8}$$

$$\underline{\sigma}\left(\frac{\partial \phi_N}{\partial \tilde{u}}(x, \hat{u}^M(x))\right) \geq r, \tag{9}$$

where $\underline{\sigma}(B)$ is the smallest singular value of the matrix B.

It is worth noting that the condition (8) is much weaker than the controllability of the linearized system. Indeed, it implies the controllability of the linearized system only at specific points of $\Omega_x \times \Omega_u^N$. The condition (9) means roughly that singularities are absent, and becomes rather technical when it is satisfied at the origin.

In fact, the assumption really needed in the following proofs is a relaxed version of Assumption 4.1 (see Remark 4.1).

It can easily be seen that the condition (8) of Assumption 4.1 enables us to apply the implicit functions theorem at any point $(x, \hat{u}^M(x)) \in \Omega_x \times U_N^\infty(x)$, while the condition (9) makes it possible to find sizes of the neighborhoods that hold *uniformly* over all possible couples $(x, \hat{u}^M(x))$ such that singular behaviors are excluded. More precisely, we have the following proposition.

Proposition 4.1. Let $x_0 \in \Omega_x$ and let $\epsilon > 0$ be given. Under Assumptions 3.1 and 4.1, there exists $\eta_\epsilon > 0$ s.t., $\forall x \in \bar{B}(x_0, \eta_\epsilon)$, $\forall M \geq N$, $\exists u(x_0, x, M)$ s.t.

$$x(N; x; u(x_0, x, M)) = x(N; x_0; \hat{u}^M(x_0)), \quad (10)$$

$$|J_N^M(x, u(x_0, x, M)) - \hat{J}_N^M(x_0)| < \epsilon. \quad (11)$$

Note that the key point of this proposition is that (11) holds for the same η_ϵ and for all $M \geqslant N$, because the trajectories $x(k; x_0; \hat{u}^M(x_0))$ and $x(k; x; u(x_0, x, M))$ becomes identical as soon as k become greater than N (according to (10)). The possibility of making the difference in (11) as small as required proceeds from continuity arguments, while the regularity assumption assumed through (9) makes it possible to find η_ϵ independent of M. (For a complete proof see Alamir and Bornard (1993).)

The following proposition gives existence results for the finite-horizon case $M < \infty$.

Proposition 4.2. Under Assumption 3.1, the problem $P_N^M(x)$ has a solution for all $x \in \Omega_x$ and all $M \geq N$. If, in addition, Assumption 4.1 holds then $\hat{J}_N^M(x)$ is continuous on Ω_x.

Proof. $P_N^M(x)$ can be written as the problem of minimizing the continuous function $J_N^M(x, \tilde{u})$ over the compact set

$$\mathcal{U}_{\text{ad}} = \{\tilde{u} \in U_{\text{ad}}^M(x) \mid J_N^M(x, \tilde{u}) \leq J_N^M(x, \tilde{u}^*)\},$$

where $\tilde{u}^* \in U_{\text{ad}}^M(x)$. Continuity proceeds from the continuity of $J_N^M(.,.)$, the compactness of the admissible space $\Omega_x \times \Omega_u^N$, and (11). □

4.2. *Study in the large.* In this subsection our aim is to prove that, for all chosen neighborhoods of the origin $\bar{B}(0, \epsilon)$, there exists a finite integer $M \geq N$ such that the finite-horizon strategy defined by (5) forces the state of the system to reach $\bar{B}(0, \epsilon)$ after a finite number of steps.

The idea is quite simple. Indeed, in the infinite-horizon case, when at instant k the infinite sequence $(\hat{u}_i^\infty(x_{k-1}))_{i \geq 1}$ is applied to the system, the cost of the resulting trajectory is exactly

$$\hat{J}_N^\infty(x_{k-1}) - (\|x_{k-1}\|_Q^2 + \|\hat{u}_0^\infty(x_{k-1})\|_R^2).$$

Nothing is to be added because the horizon is infinite.

In the case of a finite prediction horizon $M \geq N$ the cost of the final state,

$$\|x(M+1; x_{k-1}; \hat{u}^M(x_{k-1}))\|_Q^2 + \|\hat{u}_N^M(x_{k-1})\|_R^2,$$

must be added. Therefore all we have to do is to prove that, under certain conditions, and for M sufficiently large, this last cost is still sufficiently small so as to guarantee the decrease of the optimal value, *whenever x_{k-1} is still outside $\bar{B}(0, \epsilon)$.* That is the aim of the following lemma.

Lemma 4.1. Under Assumptions 3.1 and 4.1, for all $\epsilon > 0$, there exists $M_\epsilon \geq N$ such that, for all $M \geq M_\epsilon$ and all $x \in \Omega_x$,

$$\|x(M+1; x; \hat{u}^M(x))\|_Q^2 + \|\hat{u}_N^M(x)\|_R^2 < \epsilon. \quad (12)$$

Proof. Note first of all that the sequence $(\hat{J}_N^M(x))_{M \geq N}$ is increasing and upper-bounded by $\hat{J}_N^\infty(x)$ for all $x \in \Omega_x$. Therefore it is convergent. Thus

$$\lim_{M \to \infty} [\hat{J}_N^{M+L}(x) - \hat{J}_N^M(x)] = 0 \quad (13)$$

uniformly in L. This enables us to define for a given $x \in \Omega_x$, $\epsilon > 0$, the integer $\hat{M}(x, \epsilon)$ as follows:

$$\hat{M}(x, \epsilon) = \min_{M_0 \geq N} \{M_0 : \forall M \geq M_0, \forall L \geq 0, \; \hat{J}_N^{M+L}(x) - \hat{J}_N^M(x) < \epsilon\}$$

We shall prove that

$$\sup_{x \in \Omega_x} \hat{M}(x, \epsilon) < \infty \quad (14)$$

To do this, suppose that (14) is false; then one can construct two sequences of integers, $(M_i)_{i \geq 0}$, $(L_i)_{i \geq 0}$, and a sequence $(x_i)_{i \geq 0}$ of points in Ω_x such that

$$\lim_{i \to \infty} x_i = x_f \in \Omega_x, \quad (15)$$

$$\hat{J}_N^{M_i + L_i}(x_i) - \hat{J}_N^{M_i}(x_i) \geq \epsilon \quad \forall i \geq 0. \quad (16)$$

Let $M_f := \hat{M}(x_f, \frac{1}{4}\epsilon)$; then

$$\forall L \geq 0, \; \forall M \geq M_f, \quad \hat{J}_N^{M+L}(x_f) - \hat{J}_N^M(x_f) < \tfrac{1}{4}\epsilon. \quad (17)$$

On the other hand, according to (11), in a sufficiently small neighborhood of x_f,

$$\forall M \geq N, \quad \hat{J}_N^M(x) < \hat{J}_N^M(x_f) + \tfrac{1}{4}\epsilon. \quad (18)$$

Finally, by continuity of $\hat{J}_N^{M_f}(.)$, in a sufficiently small neighborhood of x_f,

$$|\hat{J}_N^{M_f}(x) - \hat{J}_N^{M_f}(x_f)| < \tfrac{1}{4}\epsilon. \quad (19)$$

Taking i sufficiently high to have x_i in the above-mentioned neighborhoods, with $M_i \geq M_f$, we obtain, according to (17)–(19),

$$\begin{aligned}
&\hat{J}_N^{M_i+L_i}(x_i) \\
&< \hat{J}_N^{M_i+L_i}(x_f) + \tfrac{1}{4}\epsilon, && \text{according to (18)},\\
&< \hat{J}_N^{M_f}(x_f) + \tfrac{1}{2}\epsilon, && \text{according to (17)},\\
&< \hat{J}_N^{M_f}(x_i) + \tfrac{3}{4}\epsilon, && \text{according to (19)},\\
&< \hat{J}_N^{M_i}(x_i) + \tfrac{3}{4}\epsilon, && \text{because } M_i \geqslant M_f.
\end{aligned}$$

The last inequality clearly contradicts (16), and therefore completes the proof of (14).

The inequality (14) means that the convergence in (13) is uniform over Ω_x. This implies in particular that

$$\lim_{M \to \infty} \left[\sum_{k=M+1}^{M+L} \|x(k; x; \hat{u}^{M+L}(x))\|_Q^2 + \|\hat{u}_k^{M+L}(x)\|_R^2 \right] = 0$$

uniformly in both L and x over Ω_x. Therefore

$$\lim_{M \to \infty} [\|x(M; x; \hat{u}^M(x))\|_Q^2 + \|\hat{u}_k^M(x)\|_R^2] = 0$$

uniformly over Ω_x. This, with $f(0,0) = 0$ and the continuity of f, complete the proof. □

Remark 4.1. It follows from the above proof that there is no need to have (11) satisfied for all $M \geq N$, but only for a subsequence $(M_i)_{i \geq 0}$. That is why Assumption 4.1 can be relaxed by requiring (8) and (9) to be satisfied on a subsequence $(M_i)_{i \geq 0}$ of prediction horizons.

We can now state the main result of this subsection.

Theorem 4.1. Under Assumptions 3.1 and 4.1, for all $\epsilon > 0$, there exists a finite $M_0 \geq N$ such that the neighborhood $\bar{B}(0, \epsilon)$ is a global attractor (over Ω_x) for the closed-loop dynamic defined by the feedback $h^M(.)$ for all $M \geq M_0$.

Proof. Let $(x_k)_{k \geq 0}$ denote the trajectory of the closed-loop system. We shall prove that

$$M_0 = \sup_{x \in \Omega_x} \hat{M}(x, \tfrac{1}{2}\epsilon) \quad (20)$$

satisfies the theorem. Indeed, for all $k \geq 0$ and all $M \geq N$,

$$\begin{aligned}
\hat{J}_N^M(x_k) \leq \hat{J}_N^M(x_{k-1}) &- \|x_{k-1}\|_Q^2 \\
&+ \|x(M+1; x_{k-1}; \hat{u}^M(x_{k-1})\|_Q^2 \\
&+ \|\hat{u}_N^M(x_{k-1})\|_R^2,
\end{aligned}$$

which (according to Lemma 4.1) gives, for all $M \geq M_0$,

$$\hat{J}_N^M(x_k) \leq \hat{J}_N^M(x_{k-1}) - \|x_{k-1}\|_Q^2 + \tfrac{1}{2}\epsilon. \quad (21)$$

This proves that each time x_{k-1} is outside $\bar{B}(0, \epsilon)$, the next step decreases the optimal value by at least $\frac{1}{2}\epsilon$. □

Before considering the local study, we shall prove the following lemma.

Lemma 4.2. Under Assumptions 3.1 and 4.1,

$$\hat{J}_N^\infty(x, \tilde{u}_{\mathrm{acc}}) = \hat{J}_N^\infty(x), \tag{22}$$

$$\hat{J}_N^\infty(x) = \lim_{m\to\infty} \hat{J}_N^M(x), \tag{23}$$

where $\tilde{u}_{\mathrm{acc}}$ is any accumulation point for the sequence $(\hat{u}^M(x))_{M\geq N}$.

Proof. Let $(\hat{u}^{M_i}(x))_{i\geq 0}$ be the subsequence that tends to $\tilde{u}_{\mathrm{acc}}$. We have, by definition

$$J_N^\infty\left(x, \lim_{i\to\infty} \hat{u}^{M_i}(x)\right) = \lim_{b\to\infty}\lim_{i\to\infty} J_N^b(x, \hat{u}^{M_i}(x))$$

$$\leq \lim_{b\to\infty}\lim_{i\to\infty} J_N^{M_i}(x, \hat{u}^{M_i}(x))$$

$$\leq \lim_{i\to\infty} \hat{J}_N^{M_i}(x) \leq \hat{J}_N^\infty(x),$$

which gives (22), because of the definition of $\hat{J}_N^\infty(x)$.

Suppose now that (23) is false; then one can find an integer p_0 and a positive real $r > 0$ s.t.

$$\forall p \geq p_0,\ \forall M \geq N,\quad J_N^p(x, \hat{u}^\infty(x)) \geq \hat{J}_N^M(x) + r. \tag{24}$$

According to (22), $\exists p_1 > p_0$ s.t.

$$\forall p \geq p_1,\quad |J_N^p(x, \hat{u}^\infty(x)) - J_N^p(x, \tilde{u}_{\mathrm{acc}})| < \tfrac{1}{4}r. \tag{25}$$

Finally, by continuity, we can find for all $p > p_1$ an integer i_p s.t.

$$\forall i \geq i_p,\quad |J_N^p(x, \tilde{u}_{\mathrm{acc}}) - J_N^p(x, \hat{u}^{M_i}(x))| < \tfrac{1}{4}r. \tag{26}$$

Choose $p > p_1$, $i > i_p$ and $M_i > p$; then

$$J_N^p(x, \hat{u}^\infty(x)) \leq \hat{J}_N^{M_i}(x) + \tfrac{1}{2}r, \tag{27}$$

which contradicts (24). □

4.3. *Local study and the main result.* The following proposition is a quite intuitive result that is necessary to prove the main result of this paper.

Proposition 4.3. Under Assumptions 3.1 and 4.1 and the N-controllability of the linearized system at the origin, there is a finite $M \geq N$ s.t. the closed-loop system defined by the feedback $h^M(.)$ is locally asymptotically stable.

Proof. Note first that one can always choose the neighborhood of the origin sufficiently small so as to have the optimal solution in the interior of the admissible region. Therefore the variation of the optimal value over the closed-loop trajectory $(x_k)_{k\geq 0}$ can be given by

$$\hat{J}_N^M(x_k) - \hat{J}_N^M(x_{k-1}) = -x_{k-1}^{\mathrm{T}} Q_M x_{k-1} + O_M(\|x_{k-1}\|^3),$$

where $-x_{k-1}^{\mathrm{T}} Q_M x_{k-1}$ is the variation of the optimal-value function corresponding to the associated LQ problem. It is clear that, under the N-controllability of the linearized system at the origin, the conditions of Lemma 4.2 are locally satisfied for the LQ problem. This enables us to write

$$\lim_{M\to\infty} Q_M = Q_\infty > 0. \tag{28}$$

If we choose M_0 s.t.

$$\forall M \geq M_0,\quad \underline{\sigma}(Q_M) > \tfrac{1}{2}\underline{\sigma}(Q_\infty) > 0 \tag{29}$$

and $\epsilon > 0$ s.t.

$$\forall x \in B(0, \epsilon),\quad \|O_M(\|x\|^3)\| < \tfrac{1}{2}x^{\mathrm{T}} Q_M x \tag{30}$$

then we have, for all $x_{k-1} \in B(0, \epsilon)$,

$$\hat{J}_N^M(x_k) \leq \hat{J}_N^M(x_{k-1}) - \tfrac{1}{4}\underline{\sigma}(Q_\infty)\,\|x_{k-1}\|^2. \tag{31}$$

This proves that each time the closed-loop trajectory passes outside $B(0, \beta)$, the optimal value decreases by at least $\frac{1}{4}\underline{\sigma}(Q_\infty)\,\|\beta\|^2$. □

We can now state the main result of this paper, which is a direct consequence of Theorem 4.1 and Proposition 4.2.

Theorem 4.2. If Assumptions 3.1 and 4.1 hold and if, in addition, the linearized system at (0, 0) is N-controllable then there exists a finite prediction horizon $\bar{M} \geq N$ such that the closed-loop system defined by the feedback law $h^M(.)$ is globally asymptotically stable for all $M \geq \bar{M}$.

5. *Conclusions*

The feasibility and the stability properties of an infinite prediction horizon and finite control horizon have been studied. Sufficient conditions that make it possible to truncate the prediction horizon while preserving the stability have been proposed. The result only states the existence of such a finite prediction horizon—its computation—remains difficult. This makes the result on truncation of a purely theoretical nature for the moment.

Alamir and Bornard (1993) proved the above results in a reference model tracking scheme and with more general cost functions. The stability results hold (as well as the arguments used in the proofs) whenever the reference model is asymptotically rapidly convergent—in particular, exponentially—and the cost function is proper and positive definite. In this paper only the main idea has been presented in order to simplify the notation.

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