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On the stability of receding horizon control of nonlinear discrete-time systems

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Abstract: In this paper, we give sufficient conditions that guarantee the existence of the receding horizon control and the stability of the feedback system. The controlled system is a general nonlinear discrete-time system. Local and global stability are studied separately, giving rise to two sets of sufficient conditions. The context is that of regulation and tracking with internal model control scheme.

Keywords: Stability; nonlinear discrete-time systems; receding horizon control.

1. Introduction

Receding horizon control strategy has been largely studied in the case of linear systems. It enables one to perform approximate tracking with relatively simple computations; furthermore, it often seems to be quite robust and, unlike the geometric approach, it is quite independent of the structural properties of the system. The stability of the feedback system has been studied in [8, 2]; it has been proved in the unconstrained case. One can find in [1] a survey of what has been done in the linear case.

In the case of nonlinear systems, only a few works exist. In [4], the problem has been studied for a time-varying discrete system, and global stability is obtained under certain assumptions. In [6, 7], the problem is examined in the case of nonlinear continuous systems, sufficient conditions for local or global stability, which are rather strong (unicity of the optimal control, continuity of this control in the initial state, strong assumption on the evolution function on the whole state space), are given.

In this paper, we go back to nonlinear discrete-time systems, in the context of internal model control with a penality on the control variations. This makes the formulation different from that used in [4].

In [4], the criterion is defined on the control and the output. This makes it necessary to make assumptions on the observability of the system; however, the state is still needed to calculate the feedback. In this paper, we define the criterion on the state and the control, and this enables us to make assumptions that are more concerned with the definition of the criterion than with the system itself which is clearly more easy to control. In particular, assumption 2.3 of [4] can be relaxed by means of assumption H′4 (see below).

This paper is organized as follows. Section 2 gives some definitions and notations. Section 3 concerns the local stability: first, the assumptions are presented, then some earlier results are recalled, and finally the main results follow. In Section 4, stability in the large is examined, the new assumptions are presented first, then, the main results.

2. Definitions and notations

2.1. The system

We consider a discrete-time system whose evolution is given by the following equations:

$$\begin{aligned} x_m(k+1) &= f_m(x_m(k), x_I(k), u(k)), \\ x_I(k+1) &= x_I(k), \\ y_m(k) &= h_m(x_m(k)), \end{aligned} \tag{1}$$

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where $x_m \in \mathbb{R}^{n_m}$ is the state vector of the system built up from a nominal system and, eventually, a stable perturbation model system, $x_I \in \mathbb{R}^{n_I}$, is a constant vector which groups together the inputs – supposed constant – of both the perturbation model and the tracking model given by:

$$\begin{aligned} x_t(k+1) &= f_t(x_t(k), x_I(k)), \\ y_t(k) &= h_t(x_t(k)), \end{aligned} \tag{2}$$

where y_t and y_m are of the same dimension p, $u \in \mathbb{R}^m$, $x_I \in \mathbb{R}^{n_I}$, $x_t \in \mathbb{R}^{n_t}$.

Equations (1) and (2) enable us to define the augmented system on which our study is based. In fact, if we consider the following definitions:

$$x(k) = \begin{pmatrix} x_m(k) \\ x_t(k) \\ x_I(k) \end{pmatrix},$$

$$f(x(k), u(k)) = \begin{pmatrix} f_m(x_m(k), x_I(k), u(k)) \\ f_t(x_t(k), x_I(k)) \\ x_I(k) \end{pmatrix}$$

$$h(x(k)) = h_t(x_t(k)) - h_m(x_m(k)),$$

we obtain the augmented system:

$$\begin{aligned} x(k+1) &= f(x(k), u(k)), \\ y(k) &= h(x(k)). \end{aligned} \tag{3}$$

In the sequel, we use the following notations:

$$z_k = (x_k, u_{k-1}),$$

$$\pi_m(x) = x_m,$$

$$\pi_t(x) = x_t,$$

$$\pi_I(x) = x_I,$$

$$n = n_m + n_t + n_I$$

2.2. The criterion

We consider a criterion $J_N(z_0, \tilde{u})$ depending on the initial state, the last vector of control applied to the system, and the sequence of the $N+1$ control vectors to be applied ($\tilde{u} \in \mathbb{R}^{(N+1)m}$). We also consider the final condition:

$$\sigma_N : \begin{Bmatrix} f(x(N; \pi_x(z_0); \tilde{u}), \tilde{u}_N) = x(N; \pi_x(z_0); \tilde{u}) \\ h(x(N; \pi_x(z_0); \tilde{u})) = 0 \end{Bmatrix},$$

where $x(N; \pi_x(z_0); \tilde{u})$ is the state reached when applying the N first vectors of $\tilde{u}$ starting from the initial state z_0, $\tilde{u}_N$ is the last vector in the series $\tilde{u}$. This condition means that $\tilde{u}$ leads to a final steady state which corresponds to a zero tracking error.

We now define the problem:

$$P(z_0) : \left\{ \min_{\tilde{u}} J_N(z_0, \tilde{u}) \,|\, \sigma_N \right\} \tag{4}$$

We note that $\hat{u}(z_0) \in \mathbb{R}^{(N+1)m}$ is the solution of (4), $\hat{x}(z_0) \in \mathbb{R}^{(N+1)n}$ is the optimal trajectory, and $\hat{J}_N(z_0) = J_N(z_0, \hat{u}(z_0))$ is the optimal value of the criterion.

2.3. Receding horizon control strategy

In this strategy, the control is defined as follows:

$$u(k) = \hat{u}_0(z(k)), \tag{5}$$

where $\hat{u}_0(z(k)) \in \mathbb{R}^m$ is the first vector of $\hat{u}(z(k)) \in \mathbb{R}^{(N+1)m}$.

The control being defined, one obtains a closed-loop system verifying the equation:

$$\begin{aligned} z(k+1) &= \begin{pmatrix} f(\pi_x(z_k), \hat{u}_0(z_k)) \\ \hat{u}_0(z_k) \end{pmatrix}, \\ &= \hat{f}(z_k). \end{aligned} \tag{6}$$

The aim of this work is to find sufficient conditions for this strategy to be possible to execute, and such that its application enables us to meet the control requirements. These requirements can be split into two: The first is to have asymptotically zero error tracking. The second is to have a transient dynamic as close as possible to that of the tracking model, avoiding rough variations on the control.

3. Local stability

In this section, we give sufficient conditions for the strategy to be executable, and for the closed-loop system (6) to be locally asymptotically stable. We first give the assumptions, then recall some earlier results, and finally give the main results which lead to conclusion.

3.1. Assumptions

In the sequel, we need the following assumptions:

(H1) $\forall x_I \in \mathbb{R}^{n_I}$, $\exists! \begin{pmatrix} x_{st} \\ u_{st} \end{pmatrix} \in \mathbb{R}^{n+m}: h(x_{st}) = 0;$

$f(x_{st}, u_{st}) = x_{st};\ \pi_I(x_{st}) = x_I.$

(H2) f and h are continuous.

(H3) $\forall x_0 \in \mathbb{R}^n$, $\exists \tilde{u}^* \in \mathbb{R}^{(N+1)m}: x(N, x(0), \tilde{u}^*)$

$= x_{st}(\pi_I(x(0))).$

(H4) The criterion is given by:

$$J_N(z_0, \tilde{u}) = \sum_{k=0}^{N} \mathscr{G}(x_k) + \mathscr{M}(\Delta\tilde{u}_k), \tag{7}$$

where $x_k = x(k; \pi_x(z_0), \tilde{u})$ and $\Delta\tilde{u}_k = \tilde{u}_k - \tilde{u}_{k-1}$. Furthermore, the following properties are satisfied:
(a) For the entire sequence of control vectors $(u_k)_{k \ge 0}$ and the corresponding trajectory $(x_k)_{k \ge 0}$, we have the following property *whenever the trajectory is bounded*:
$\mathscr{G}(x_k) + \mathscr{M}(\Delta u_k) \to 0$ implies $(x_k, u_k) \to \{(x_{st}(\pi_I(x_0)), u_{st}(\pi_I(x_0)))\}$
(b) $\mathscr{M}(u) \to \infty$ when $\|u\| \to \infty$
(c) $\mathscr{M}$ is continuous positive-definite.
(d) $\mathscr{G}(x - x_{st})$ is continuous positive-definite.

(H5) $\forall x_I \in \mathbb{R}^{n_I}$, the optimal solution is such that $\hat{u}(z)$ is continuous at $z_{st}(x_I)$ in the topology induced on $\mathbb{R}^{n+m} \cap \{z: \pi_I(\pi_x(z)) = x_I\}$.

Remark 3.1. Assumptions (H1) and (H3) include necessary conditions for the feasibility of the regulation by the proposed strategy, for all possible inputs to the reference model and all initial states of the system. It can be simplified if the input x_I takes only certain possible values.

As for the unicity of (x_{st}, u_{st}), it is not essential and can be replaced by a choice of a candidate couple that satisfies (H3), and then, a criterion with the chosen one such that (H4.a) and (H4.b) hold can be built up.

Remark 3.2. Assumption (H3) implies, in particular, that the reference model reaches its steady state after a finite number of steps. It excludes, for example, exponential evolutions.

Remark 3.3. Assumption (H4.d) is simply technical. Indeed, if one takes:

$\mathscr{G}(x) = h^T(x).G.h(x) + \lambda.\|x - x_{st}\|$

with $\lambda > 0$ infinitely small, then both (H4.d) and the tracking requirement are respected.

3.2. Recalls

In order to demonstrate the stability, we use the following theorem which is a slightly modified version of the one given in [3] which is a discrete formulation of Lasalle's theorem. This theorem will be used to demonstrate both local and global stability.

Fundamental theorem. *Let G be a subset of the state space. If we can find a function $V(z)$ defined on G, bounded below such that*

$V(\hat{f}(z_k)) - V(z_k) \le -W(z_k) \le 0,$

and if in addition, we have (the property):

$W(z_k) \to 0$ implies $z_k \to \mathscr{D}$ whenever the trajectory is bounded,

then, for all trajectories of the closed-loop system $z_{k+1} = \hat{f}(z_k)$ which remain in G, only two situations are possible:
- *Either the trajectory is unbounded.*
- *Or it is bounded and then z_k tends to $\mathscr{D}$ when $k \to \infty$*

Remark 3.4. In [3], the continuity of W is required. Actually, this is a particular case of the theorem. In this case the set $\mathscr{D}$ is an invariant subset of $\mathscr{A}$, where $\mathscr{A}$ is the set defined by:

$\mathscr{A} = \{z \in \bar{G} \mid W(z) = 0\}.$

Proof of the fundamental theorem. The sequence $V(z_k)$ is nonincreasing and bounded below, so it must approach a limit as $k \to \infty$. However, we have by assumption: $V(\hat{f}(z_k)) - V(z_k) \le -W(z_k) \le 0$. Thus, we must have $W(z_k) \to 0$. The end of the proof is straightforward. □

3.3. Local stability: main results

In this subsection, we first prove the existence of the solution and, consequently, that the strategy is realizable. After that, we prove that $\hat{J}_N(z)$ is a Lyapunov function – meeting the requirements of the fundamental theorem. It remains to be proved that the trajectory of the closed-loop system is bounded, to be able to give the theorem on the local stability which ends this section.

Proposition 3.1. *If Assumptions* (H1)–(H4) *are satisfied, then* $\forall z_0 \in \mathbb{R}^{n+m}$, *the Problem* $\mathrm{P}(z_0)$ *has a solution.*

Proof.

- Let $U_{ad}(z_0) \overset{\text{def}}{=} \{\tilde{u} \in \mathbb{R}^{(N+1)m}:$

$$h(x(N; \pi_x(z_0); \tilde{u})) = 0,\ f(x(N; \pi_x(z_0); \tilde{u}), \tilde{u}_N)$$
$$= x(N; \pi_x(z_0); \tilde{u}),\ J_N(z_0, \tilde{u}) \le J_N(z_0, \tilde{u}^*)\},$$

$\tilde{u}^*$ being that defined in Assumption (H3).

- The problem $\mathrm{P}(z_0)$ can be written equivalently:

$$\mathrm{P}(z_0) = \underbrace{\min}_{\tilde{u} \in U_{ad}(z_0)} J_N(z_0, \tilde{u})$$

- According to assumptions (H1) and (H3), this set is nonempty. Furthermore, one can prove that it is compact. In fact:
  - $U_{ad}(z_0)$ is clearly closed, being the inverse image of a closed set by a continuous function [(H2)(H4)].
  - It is bounded because of (H4.b) and the upper bound is imposed on the criterion in the definition of $U_{ad}(z_0)$. Hence, this set is compact.
- $J_N(z_0, \tilde{u})$ is continuous in u; therefore, it achieves its minimum on the compact $U_{ad}(z_0)$. □

Proposition 3.2. *If assumptions* (H1)–(H4) *are satisfied, the functions*

$$V(z) = \hat{J}_N(z),$$

$$W(z) = \mathscr{G}(\pi_x(z)) + \mathscr{M}(\hat{u}_0(z) - u_{-1})$$

satisfy the requirements of the fundamental theorem with $\mathscr{D} = \{(x_{st}(x_0), u_{st}(x_0))\}$.

Proof.

- Let

$$z_k = \begin{pmatrix} x_k \\ u_{k-1} \end{pmatrix}$$

be the state of the closed-loop system at the instant k, let $\hat{u}(z_k)(\in \mathbb{R}^{(N+1)m})$ be the corresponding optimal control and $\hat{z}(z_k)$ the optimal trajectory. Then $\hat{z}_1(z_k)$ is the point that follows z_{is}.

- Define, for the point $\hat{z}_1(z_k)$, a suboptimal control $\bar{u}(\hat{z}_1(z_k))$ as follows:

$$\bar{u}_j(\hat{z}_1(z_k)) = \begin{cases} \hat{u}_{j+1}(z_k), & \text{if } j < N, \\ \hat{u}_N(z_k), & \text{if } j = N. \end{cases}$$

- Let us compute the variation of $V(z)$ along the trajectory of the closed-loop system:

$$\begin{aligned} \Delta V(z_k) &= V(z_{k+1}) - V(z_k) \\ &= \hat{J}_N(z_{k+1}) - \hat{J}_N(z_k) \\ &\le \bar{J}_N(z_{k+1}) - \hat{J}_N(z_k) \\ &= \underbrace{-\mathscr{G}(\pi_x(z_k)) - \mathscr{M}(\Delta\hat{u}_0(z_k))}_{-W(z_k)} \end{aligned}$$

The terms due to the suboptimal control vanish by definition of the problem $\mathrm{P}(z_{k+1})$.

- V is clearly bounded below by 0.
- Suppose that $W(z_k) \to 0$ and the trajectory is bounded, then we have according to assumption (H4.a):

$$(x_k, u_k) \to \{(x_{st}, u_{st})\} = \mathscr{D} \qquad \square$$

Proposition 3.3. *If assumptions* (H1)–(H5) *are satisfied, then*

$\forall x_I \in \mathbb{R}^{n_I}$ *(thus* $z_{st}(x_I)$*) the trajectory of the closed-loop system is locally bounded (around the point* z_{st}*).*

More precisely: $\exists \varepsilon > 0$ *such that* $\forall z_0 \in B(z_{st}, \varepsilon) \cap \{z \mid \pi_I(\pi_x(z)) = x_I\}$, *one has*

The trajectory of the closed-loop system starting from z_0 *is bounded.*

Proof.

- Define $B_{x_I}(z_{st}, \varepsilon_i)$ by $B_{x_I}(z_{st}, \varepsilon_i) = B(z_{st}, \varepsilon_i) \cap \{z \mid \pi_I(\pi_x(z)) = x_I\}$.
- According to (H5), there exist $r > 0$ and $\varepsilon_1 > 0$ such that:

$$\forall z \in B_{x_I}(z_{st}, \varepsilon_1) \quad \hat{u}_0(z) \in B(u_{st}, r).$$

- The function f is continuous on the compact set $S_z = \pi_x(\bar{B}_{x_I}(z_{st}, \varepsilon_1)) \times \bar{B}(u_{st}, r)$. Therefore,

$$\varepsilon_2 \overset{\text{def}}{=} \underbrace{\max}_{z \in \bar{B}_{x_I}(z_{st}, \varepsilon_1)} \|\hat{f}(z) - z_{st}\|$$

is well defined.

- Either $\varepsilon_2 \le \varepsilon_1$, in which case, the proof is achieved; or $\varepsilon_2 > \varepsilon_1$, in which case, further development is necessary for giving the proof.

- Consider the compact set $\bar{\sigma} = \bar{B}_{x_I}(z_{st}, \varepsilon_2) - B_{x_I}(z_{st}, \varepsilon_1)$. On this compact set, the function $\hat{J}_N(z)$ satisfies:

$$\hat{J}_N(z) > J_0 \neq 0 \quad \forall z \in \bar{\sigma}.$$

  This is because $\mathscr{G}(x - x_{st})$, and $\mathscr{M}$ are, both, positive-definite.
- According to (H5), $\hat{J}_N(z)$ is continuous at z_{st}, and vanishes at z_{st}. Therefore, $\exists \varepsilon_3 > 0$ such that

$$\hat{J}_N(z) < J_0 \text{ for all } z \in B_{x_I}(z_{st}, \varepsilon_3)$$

- Let $z_0 \in B(z_{st}, \varepsilon_3)$. By definition of ε_3, none of the points of the closed-loop system's trajectory is in $\bar{\sigma}$. On the other hand, starting from a point in $B(z_{st}, \varepsilon_1)$, the trajectory cannot leave $B(z_{st}, \varepsilon_2)$. This shows that, starting from any point in $B(z_{st}, \varepsilon_3)$, the trajectory remains in $B(z_{st}, \varepsilon_1)$. □

We can now state our result on local stability.

Theorem 1. *If Assumptions* (H1)–(H5) *are satisfied, the closed-loop system is locally asymptotically stable for all* $x_I \in \mathbb{R}^{n_I}$ *(in the same sense as that of Proposition* 3.3).

Proof. It is a direct consequence of Propositions 3.2, 3.3, and the fundamental theorem. □

4. Global stability

In this section, we give sufficient conditions that guarantee the global stability of the closed-loop system. We first present the new assumptions. Then we follow the same development as that followed in the proof of local stability. We give only the changes that one must make in the previous arguments.

4.1. Assumptions

- Assumptions (H1)–(H3) remain unchanged.
- (H'4)
  - (H4.a)–(H4.c) hold.
  - $\mathscr{G}$ is continuous nonnegative.
  - Furthermore:
    For all x_I and all compact sets $S_u \subset \mathbb{R}^{(N+1)m}$ $\exists b_{S_u}: R^+ \to R^+$ such that $b_{S_u}(a) \to \infty$ when $a \to \infty$ and

$$J_N(z_0, \tilde{u}) > b_{S_u}(\| z_0 \|)$$

$$\forall \tilde{u} \in S_u \text{ and } \forall z_0 \text{ such that } \pi_I(\pi_x(z_0)) = x_I.$$

Remark 4.1. The last assumption guarantees a kind of observability by the criterion at infinity. It clearly means that one cannot keep a finite cost value with bounded control when the initial state goes infinitely far from the desired final steady state.

4.2. Global stability: main results

The proof of the existence of the solution for the problem $\mathrm{P}(z_0)$ can be carried out exactly as before. Thus, one can state the following proposition.

Proposition 4.1. *If Assumptions* (H1)–(H3) *and* (H'4) *are satisfied, then*

$\forall z_0 \in \mathbb{R}^{n+m}$, *the Problem* $\mathrm{P}(z_0)$ *has a solution.*

It is clear that the optimal value function is still somehow a Lyapunov function which meets the requirements of the fundamental theorem. Hence we have the following proposition.

Proposition 4.2. *If Assumptions* (H1)–(H3) *and* (H'4) *are satisfied, then*

$$V(z) = \hat{J}_N(z),$$

$$W(z) = \mathscr{G}(\pi_x(z)) + \mathscr{M}(\hat{u}_0(z) - u_{-1})$$

satisfy the requirements of the fundamental theorem with $\mathscr{D} = \{(x_{st}(x_0), u_{st}(x_0))\}$.

We can finally state the main result on the global stability:

Theorem 2. *If Assumptions* (H1)–(H3) *and* (H'4) *are satisfied, then, for all* $x_I \in \mathbb{R}^{n_I}$, *the closed-loop system is globally asymptotically stable on* $\mathbb{R}^{n+m} \cap \{z \mid \pi_I(\pi_x(z)) = x_I\}$.

Proof. The trajectory of the closed-loop system is bounded because $\hat{J}_N(z_k) \leq \hat{J}_N(z_0)$ which, with the last assumption of (H'4), implies that the z_k are bounded. Once again, the second situation of the fundamental theorem holds. The set $\mathscr{D}$ being reduced to $\{z \mid \pi_x(z) = x_{st}\}$, this implies the result. □

5. Conclusion

In this paper, the existence and the stability of the receding horizon control for nonlinear discrete-time systems have been studied. Sufficient conditions for local and global stability are given separately. These conditions seem to be quite minimal; especially those of global stability. Simulations are in preparation to test the efficiency of the strategy and, eventually, its limits.

References

[1] C.E. García, D.M. Prett and M. Morari, Model predictive control: theory and practice – a Survey, *Automatica – J. IFAC* **25** (1989) 335–348.

[2] J.P. Gauthier and G. Bornard, Commande multivariable en présence de contraintes de type inégalité, *RAIRO Automat./Systems Anal. Control* **17** (1983) 205–222.

[3] J. Hurt, Some stability theorems for ordinary difference equations, *SIAM J. Numer. Anal.* **4** (1967) 582–596.

[4] S.S. Keerthi and E.G. Gilbert, Moving-horizon approximations for a general class of optimal nonlinear infinite-horizon discrete-time systems, in: *Information Sciences and Systems, Proc. 20th Ann. Conf.*, Princeton University.

[5] J.P. Lasalle, Stability theory for ordinary differential equations, *J. Differential Equations* **4** (1968) 57–65.

[6] D.Q. Mayne and H. Michalska, Receding horizon control of nonlinear systems, *IEEE Trans. Automat. Control* **35**, July (1990) 814–824.

[7] H. Michalska and D.Q. Mayne, Receding horizon control of nonlinear systems without differentiability of optimal value function, *Systems Control Lett.* **16** (1991) 123–130.

[8] Y. Thomas and A Barraud, Commande optimale à horizon fuyant, *RAIRO Automat. Systems Anal. Control n Avril* (1974) J-1, 126–140.