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A benchmark for optimal control problem solvers for hybrid nonlinear systems☆

Mazen Alamir*

LAG-ENSIEG, Rue de la Houille Blanche, BP 46, Domaine Universitaire. 38400 Saint Martin d'Hères, France

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Abstract

In this paper, a benchmark problem is proposed in order to assess comparisons between different optimal control problem solvers for hybrid nonlinear systems. The model is nonlinear with 20 states, 4 continuous controls, 1 discrete binary control and 4 configurations. Transitions between configurations lead to state jumps. The system is inspired by the simulated moving bed, a counter-current separation process.

Keywords: Hybrid Nonlinear Systems; Optimal Control; Benchmark; SMB chromatography

1. Introduction

Hybrid system modelling and control are key issues in nowadays engineering context (Branicky, Borkar, & Mitter, 1998). In particular, computing optimal control for such systems is a burning and challenging question and many algorithms have been already proposed. These algorithms are based on quite wide range of techniques including dynamic programming (Hedlund & Rantzer, 2002), mixed-integer logical or piece-wise affine approximations (Bemporad, Borrelli, & Morari, 2000), two-stages like approaches (Shahid Shaikh & Caines, 2003; Xu & Antsaklis, 2004) and maximum principle-based iterations (Alamir & Attia, 2004; Attia, Alamir, & Canudas de Wit, 2005). Controversial debates about the range, feasibility or scalability of these algorithms underline the need for a unified benchmark to assess comparison and performance characterization. This paper proposes such a benchmark. The relevance of the proposed benchmark comes from the following fact: (1) The relatively high-dimensional state vector as well as the number of configurations. This clearly makes the benchmark challenging for both dynamic programming approaches and piece-wise affine approximation-based algorithms that seem to show serious difficulties beyond four states. (2) The proposed benchmark is physically meaningful although applied mathematicians might use it in a blind manner. (3) The hybrid character of the original process is crucial. Indeed, The basic regimen of the process is cyclic and no steady state can be compatible with the control objective. This may challenge two stages approaches that seem to rely on a few number of switches in order to avoid solving many classical optimal control problems associated to each candidate switching sequence. (4) The model nonlinearities are scalable, the benchmark can then be used in purely linear, weakly nonlinear or strongly nonlinear mode. This is an interesting feature to test to which extent, linear or piece-wise affine approximations are able to tackle nonlinear problems.

The benchmark includes: controlled switches between configurations, state jumps, discrete set-valued controls. However, autonomous switches and configuration dependent dynamics are not included.

The paper is organized as follows: first the hybrid dynamical model is presented in Section 2. Section 3 tells some words about the genesis of the model. The optimal control problems as well as a basic solution for future comparisons are given in Section 4.

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* Tel.: +33 4 76826326; fax: +33 4 76826388.
E-mail address: mazen.alamir@inpg.fr.

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2. Model's definition

2.1. The basic elementary model

The benchmark is constructed by connecting together several "*copies*" of an elementary model that is described in the present section. The elementary model has 4-dimensional state $z := (z_a^T \quad z_b^T)^T$ (where $z_a \in \mathbb{R}^2$ and $z_b \in \mathbb{R}^2$) and two scalar outputs y_a and y_b. The dynamical equations for the elementary model are given by

$$\dot{z}_j = [A_j(z)]z_j + [B_j(z,\varepsilon)]e, \quad j \in \{a, b\}, \tag{1}$$

$$y = \begin{pmatrix} y_a \\ y_b \end{pmatrix} = Cz, \tag{2}$$

where $e \in \mathbb{R}^2$ is a two dimensional input vector, $\varepsilon \in \{-1, 1\}$ is a discrete control and

$$A_j(z) = \begin{pmatrix} 0 & 1 \\ -\omega_j^2 & -2\xi_j(1-\varphi_j(z))\omega_j \end{pmatrix}, \tag{3}$$

$$B_a(z,\varepsilon) = \begin{pmatrix} 0 & 0 \\ \omega_a^2 & \varepsilon \cdot \psi_a(z) \end{pmatrix}, \tag{4}$$

$$B_b(z,\varepsilon) = \begin{pmatrix} 0 & 0 \\ \varepsilon \cdot \psi_b(z) & \omega_b^2 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{5}$$

The nonlinear terms $\varphi_j(\cdot)$ and $\psi_j(\cdot)$ are degrees of freedom in the definition of the benchmark provided that they meet the following requirements

$$\forall z, \forall j \in \{a, b\}$$
$$0 \leqslant \varphi_j(z) \leqslant 1; \quad 0 \leqslant \frac{\psi_j(z)}{\omega_j^2} \leqslant 0.5$$

note that when φ_j and ψ_j are taken equal to 0, the elementary model becomes linear and decoupled. When they are taken constant, the model becomes linear and coupled while for general choices, the model is nonlinear and coupled. For instance, the following choices may be considered ($K_a, K_b \geqslant 0$)

$$\varphi_j(z) = \frac{K_j|y_j|}{1 + K_a|y_a| + K_b|y_b|}, \tag{6}$$

$$\psi_j(z) = \alpha_j \omega_j^2; \quad \alpha_j \leqslant 0.5. \tag{7}$$

2.2. Connecting elementary models

Since four elementary models are used, the state of the ith model is denoted here by $x^{(i)} \in \mathbb{R}^4$. Namely

$$x^{(i)} = \begin{pmatrix} z_a \\ z_b \end{pmatrix}\Bigg|_{\text{Elementary model } i}. \tag{8}$$

The corresponding dynamical (1) is shortly written as follows ($i \in \{1, \ldots, 4\}$):

$$\dot{x}^{(i)} = F(x^{(i)}, \varepsilon_i, e_i); \quad e_i \in \mathbb{R}^2; \quad \varepsilon_i \in \{-1, 1\}. \tag{9}$$

"*Connecting*" the elementary models amounts to make the inputs $(e_i)_{i=1}^4$ of the four elementary models depending on the extended state $x_e = \left(x^{(1)^T} \quad \ldots \quad x^{(4)^T}\right)^T \in \mathbb{R}^{16}$ as well as on some control input vector $u \in \mathbb{R}^4$ and some measured exogenous signal w. This is formally written as follows:

$$e_i = M_i(u)x_e + N_i(u)w; \quad i \in \{1, \ldots, 4\}, \tag{10}$$

where the coupling matrices $M_i(u) \in \mathbb{R}^{2\times 16}$ and the exogenous signal gain matrices $N_i(u)$ can be easily determined from the following relations:

$$e_1 = \frac{u_1 - u_4}{u_1} Cx^{(4)}; \quad e_2 = Cx^{(1)}, \tag{11}$$

$$e_3 = \frac{u_2}{u_3} Cx^{(2)} + \frac{(u_3 - u_2)}{u_3} w; \quad e_4 = Cx^{(3)}, \tag{12}$$

the control inputs $(u_i)_{i=1}^4$ have to meet the following constraints:

$$u_{\min} \leqslant \begin{pmatrix} u_2 \\ u_4 \\ u_1 - \max(u_2, u_4) \\ u_3 - \max(u_2, u_4) \end{pmatrix} \leqslant u_{\max}. \tag{13}$$

The system shows four configurations. The dynamical equations are identical whatever is the active configuration. However, when switching from some configuration q_1 to a different configuration q_2 at some instant t, a discontinuity on the state arises according to the following equations:

$$x_e(t^+) = S(q_1, q_2) \cdot x_e(t); \quad q_1, q_2 \in \{1, \ldots, 4\}, \tag{14}$$

$$S(q, q+1) = S(q, q-3) := \begin{pmatrix} 0 & \mathbb{I}_{4\times 4} & 0 & 0 \\ 0 & 0 & \mathbb{I}_{4\times 4} & 0 \\ 0 & 0 & 0 & \mathbb{I}_{4\times 4} \\ \mathbb{I}_{4\times 4} & 0 & 0 & 0 \end{pmatrix},$$
$$S(q, q+2) = S(q, q-2) := S^2(q, q+1),$$
$$S(q, q+3) = S(q, q-1) := S^3(q, q+1).$$

It is worth noting that the switches between configuration induces state jumps that are crucial in the achievement of the control objective defined later on.

In order to express the control objective (see below), two additional state vectors $\eta_1 \in \mathbb{R}^2$ and $\eta_2 \in \mathbb{R}^2$ are needed with dynamics given by

$$\dot{\eta}_1 = (u_1 - u_2) \cdot Cx^{(1)}; \quad \dot{\eta}_2 = (u_3 - u_4) \cdot Cx^{(3)}. \tag{15}$$

To summarize, the hybrid system has the 20-dimensional state $x := (x_e^T \quad \eta_1^T \quad \eta_2^T) \in \mathbb{R}^{20}$ with the following dynamics [see (9)]

$$\dot{x}^{(i)} = F(x^{(i)}, \varepsilon_i, M_i(u)x + N_i(u)w); \quad i \in \{1, \ldots, 4\},$$
$$\dot{\eta}_1 = (u_1 - u_2) \cdot Cx^{(1)},$$
$$\dot{\eta}_2 = (u_3 - u_4) \cdot Cx^{(3)},$$
$$x(t^+) = S(q(t), q(t^+)) \cdot x(t),$$

where the control input u satisfies the set of constraints (13).

3. Some words on the model's genesis

The benchmark proposed in the preceding section is inspired by the binary simulated moving bed chromatographic process (Klatt, Hanisch, & Dunnebier, 2002). This is a process enabling two different species present in the same flow (injected at inlet 2 in Fig. 1) to be partially separated. More precisely, each elementary model i (the blocs labelled **EMi** in Fig. 1) may be viewed as a separation column that receives at its inlet some fluid injected with a flow rate u_i and characterized by concentration $e_i \in \mathbb{R}_+^2$. The first [resp. second] component of e_i is the concentration of the first species (a) [resp. second species (b)] in the inlet flow. In this context, the first component of the vector z_j for $j \in \{a, b\}$ stands for the concentration of the species j in the chromatographic column while the second component represents the derivative of this concentration. The output vector's components of the elementary model ($y^{(i)}$) are the concentrations in the flow "*leaving*" the column. Different values of ω_a and ω_b simulate different adsorption affinities of the two species w.r.t the adsorbent. It is this difference in travelling speeds that enables the two species to be separated and collected at outlets 1 and 2, respectively (see Fig. 1). Note that the exogenous signal $w \in \mathbb{R}^2$ in Fig. 1 stands for the concentration of species a and b in the feed stream. Furthermore, with reference to the SMB process, a configuration is defined for a given position of the inlet/outlet ports. Switching between configurations amounts to change the position of these ports in order to better control the mean purities being achieved and leads to state jumps.

Finally, note that the terms $(u_1 - u_2) \cdot Cx^{(1)}$ [resp. $(u_3 - u_4) \cdot Cx^{(3)}$] are the mass flow rates of species a and b that are available at the outlet 1 [resp. outlet 2]. Consequently, provided that $\eta_1(0)=\eta_2(0)=(0,0)$, $\eta_1(t) \in \mathbb{R}_+^2$ and $\eta_2(t) \in \mathbb{R}_+^2$ stand for the quantity of species a and b, respectively, extracted at outlet 1 [resp. outlet 2] during the time interval $[0, t]$. In particular,

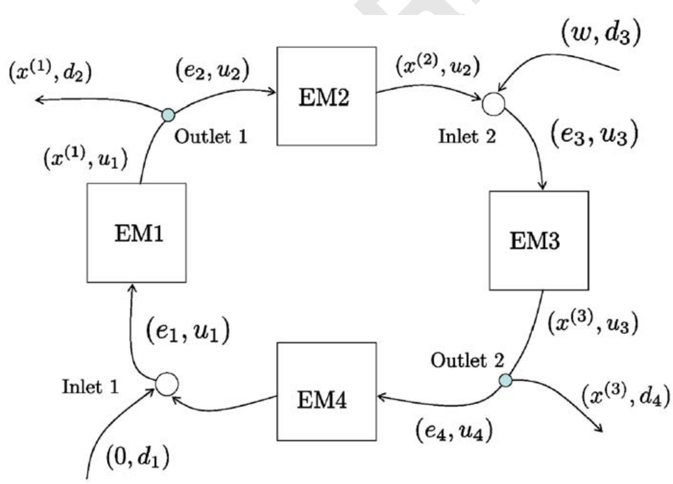

Fig. 1. Schematic view of the benchmark invoking the chromatography related genesis.

Table 1
Suggested values for the benchmark

System's parameters		Problem's parameters	
(ω_a, ω_b)	$(2, 1)$	(T, μ_1)	$(20, 0.01)$
(ξ_a, ξ_b)	$(0.7, 1.1)$	w	$(0.5, 0.5)$
(K_a, K_b)	$(0.01, 0.03)$	(p_1^d, p_2^d)	$(0.8, 0.7)$
(α_a, α_b)	$(0.01, 0.02)$	$(u_{\min}, u_{\max})$	$(0.05, 5.0)$

the quantities

$$h_1(x(t)) := \frac{x_{18}(t)}{x_{17}(t) + x_{18}(t)},$$
$$h_2(x(t)) := \frac{x_{19}(t)}{x_{19}(t) + x_{20}(t)},$$

stand for the purities of species b and a in the flow extracted on $[0, t]$ at outlets 1 and 2, respectively.

4. The optimal control problem's formulation/a sub-optimal solution

Based on the discussion of Section 3, the optimal control problem under interest may be clearly stated as follows:

$$\min_{u(\cdot), q(\cdot), \varepsilon(\cdot)} J_{(\mu_1, \mu_2)} = [h_1(x(T)) - p_1^d]^2 + [h_2(x(T)) - p_2^d]^2 + \mu_1 \int_0^T [u_1(\tau) - u_4(\tau)]^2 \, d\tau - \mu_2 [x_{18}(T) + x_{19}(T)] \quad (16)$$

under the saturation constraint (13) on u and the following initial conditions

$$x_i(0) = \begin{cases} 0.05 & \text{if } i \text{ is odd and } i \leqslant 16 \\ 0 & \text{otherwise} \end{cases} ; \quad q(0) = 1.$$

Using the analogy of Section 3, the cost function (16) may be interpreted as follows: the first two quadratic terms in (16) express the aim to obtain products that satisfy some pre-specified purities p_1^d, $p_2^d \in [0, 1]$. The third term is a production cost weighting (the quantity of the desorbent being used) while the last term expresses the aim of maximizing the yields at the outlets 1 and 3.

In the remainder of this section, basic solutions are proposed for the optimal control problem (16) that can be used in future comparisons. The system's and problem's parameters are those suggested in Table 1. as for μ_2, two values have been used, namely $\mu_2 = 0.001$ and 0.005 yielding two optimal control problems that are referred to by $P_{\mu_2=0.001}$ and $P_{\mu_2=0.005}$, respectively.

The proposed basic solutions use the following simple parametrization of control profiles $u(\cdot)$, $q(\cdot)$ and $\varepsilon(\cdot)$:

Constant continuous control $u(t) = (u_1^0, u_2^0, u_3^0, u_4^0)^T$ for all $t \in [0, T]$.

Constant discrete control $\varepsilon(t) = \varepsilon^0$ for all $t \in [0, T]$.

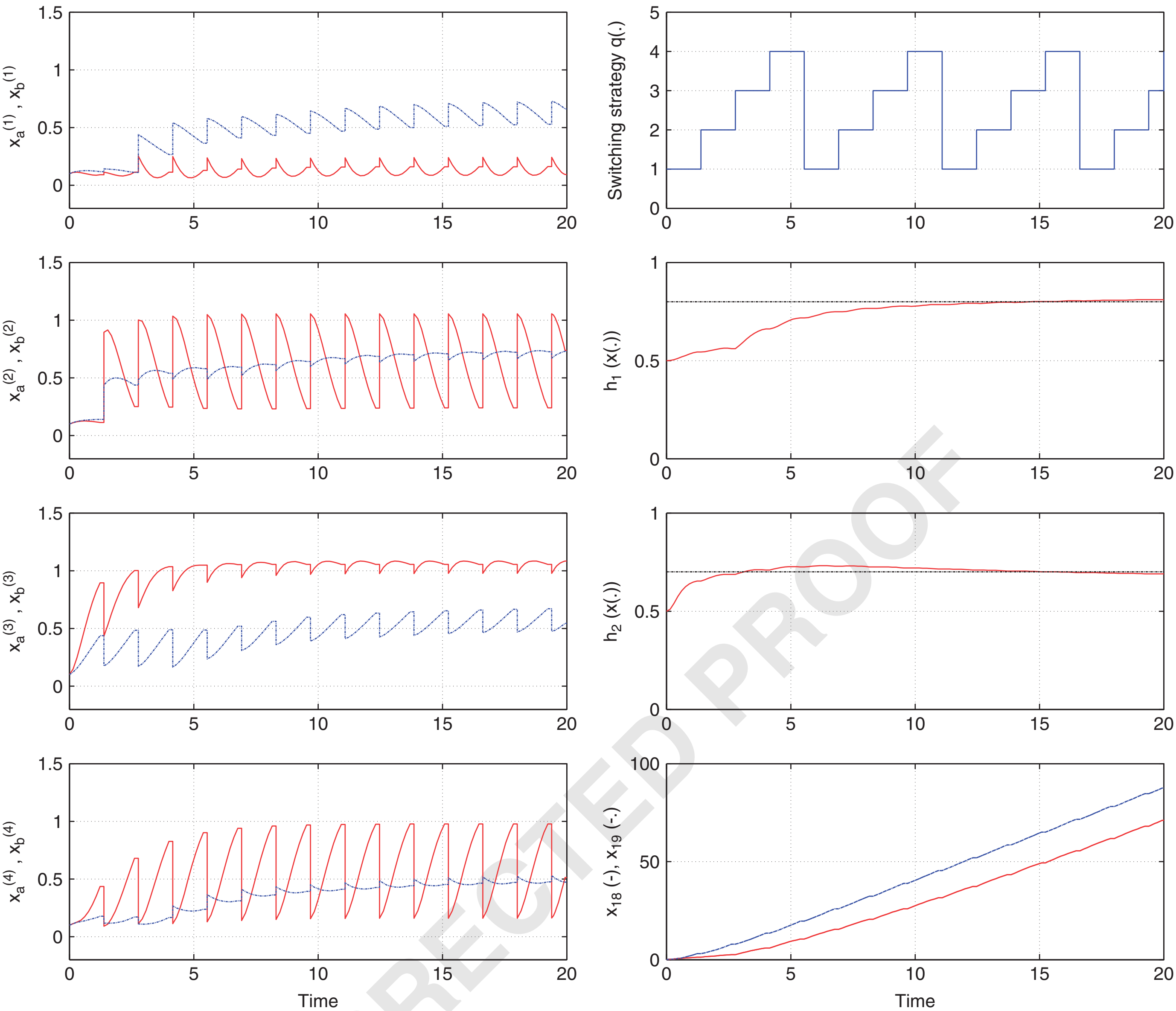

Fig. 2. The proposed solution for $P_{\mu_2=0.001}$: (left) trajectories of the four elementary models outputs; (right) from top to bottom: the switching strategy, the regulated output h_1, the regulated output h_2 and the evolution of x_{18} and x_{19} (See (17) for the sub-optimal control values).

Cyclic switching strategy with some period $\tau_s^0 > 0$, namely

$$q(t_k^+) = succ_4(q(t_k)); \quad t_k = k\tau_s^0, \tag{17}$$

where $succ_4(j) = j + 1$ for $j \in \{1, \ldots, 3\}$ and $succ_4(4) = 1$.

Based on the above simple parameterization, nonlinear programming has been used to find the optimal parameters $(u_j^0)_{j=1}^4$ and τ_s^0. This results in the following parameters:

Case $\mu_2 = 0.001$.

$$(u_1^0, u_2^0, u_3^0, u_4^0, \tau_s^0, J) = (8.05, 0.05, 10.0, 5.0, 1.39, -0.141), \tag{18}$$

Case $\mu_2 = 0.005$.

$$(u_1^0, u_2^0, u_3^0, u_4^0, \tau_s^0, J) = (10.0, 0.05, 10.0, 5.0, 1.36, -0.845). \tag{19}$$

Note that in the case $\mu_2 = 0.001$, the final values of h_1 and h_2 are almost equal to their desired values $p_1^d = 0.8$ and $p_2^d = 0.7$ (see Fig. 2). This is not the case when $\mu_2 = 0.005$ since the relative importance of this latter requirement and the "production related requirement" (the last term in the cost function) lead to increase x_{18} rather than maintain "*good*" final values for h_1. Note also that in the case $\mu_2 = 0.005$, all the components of the continuous control u are on the boundaries of the admissible region according to the constraint (13). To this respect, it would be interesting to investigate whether a more rich (non-

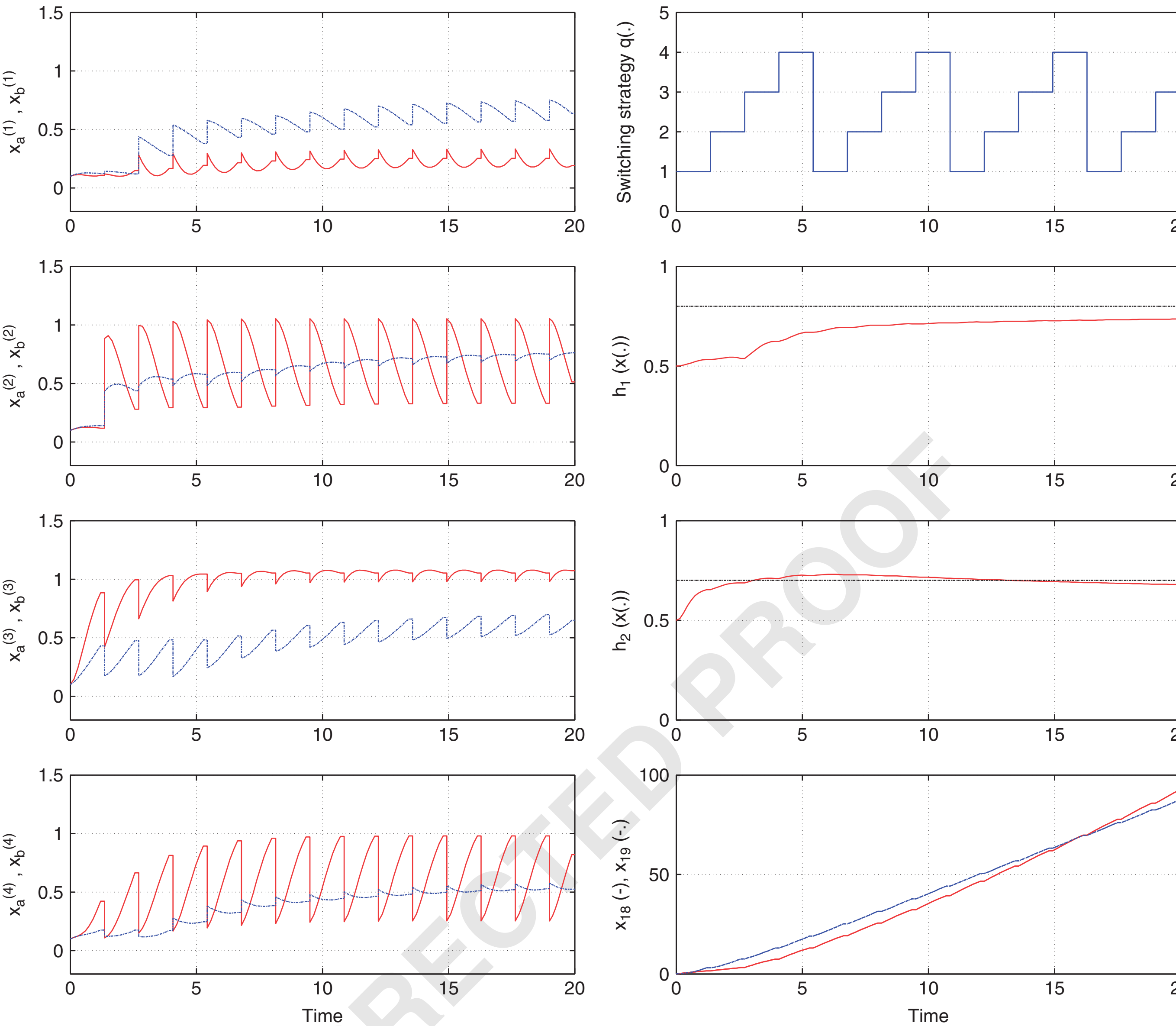

Fig. 3. The proposed solution for $P_{\mu_2=0.005}$: (left) trajectories four elementary models outputs; (right) from top to bottom: the switching strategy, the regulated output h_1, the regulated output h_2 and the evolution of x_{18} and x_{19} (See (18) for the sub-optimal control values).

constant) profiles for the continuous control u or/and a non-periodic switching strategies for the configuration index may provide solutions that recover the quality of the "*final*" state (Fig. 3).

5. Conclusion

In this paper, a benchmark is proposed for the validation of algorithms aiming to solve optimal control problems for nonlinear hybrid systems. A set of problem's parameter is proposed together with a reference solution to be used as a basis for comparison. Under certain choices of the problem's parameters ($\mu_2 = 0.005$), the performance achieved by the reference solution shows some limitations that is probably due to the over-simplified parameterization giving thus a reasonable margin for improvement using more sophisticated algorithms.

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