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Brief Paper

Discontinuous exponential stabilization of chained form systems☆

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Abstract

In this paper a novel approach to exponential stabilization of chained form system is proposed. Provided that the control law fulfills mild assumptions, it is shown that a Brunovsky-like change of coordinates depending on the control transforms any chained form system into a system having a linearly bounded nonlinear part. Using linear tools, it is possible to exponentially stabilize a chained form system for any initial condition outside a submanifold containing the origin. This result is then generalized to global exponential stabilization. The so obtained static discontinuous feedback is bounded for bounded states and exponentially converges to zero along the closed-loop trajectories of the system.

Keywords: Nonholonomic systems; Global exponential stabilization; Discontinuous feedback

1. Problem statement

The stabilization of nonholonomic systems, that is Lagrange systems with linear nonintegrable constraints, have focused a lot of attention this last years (see e.g. survey papers Kolmanovky & McClamroch, 1995; Canudas de Witt, Siciliano, & Bastin, 1996). The absence of feedback linearizing transformation (Isidori, 1995) as of continuous stabilizing static feedback law although they are open-loop controllable (Brockett, Millman, & Susmann, 1983) render nonholonomic systems rather difficult to stabilize. This probably explains, between other reasons, their popularity and more particularly the one of chained form that rapidly became prime quality systems for the development of new array of advanced control strategies. Among these strategies, let us mention the two main classes: regular time-varying feedbacks (Coron, 1992, 1995; Teel, Murray, & Walsh, 1995; Samson, 1995; Lin, 1996; Morin & Samson, 1997) and discontinuous feedbacks (Astolfi, 1996).

Chained form systems are known to make an important class of driftless nonholonomic systems including the knife-edge, the underactuated rigid spacecraft and a car towing several trailers. As mentioned by Murray and Sastry (1993) and Kolmanovky and McClamroch (1995), many nonlinear mechanical systems can be transformed via coordinates change and feedback into chained form. A driftless nonholonomic system in single chained form is described by the following differential equations (without loss of generality, we assume that $n > 1$):

$$\dot{x}_0 = u_0, \tag{1}$$

$$\dot{x} = u_0 A x + B u_1, \tag{2}$$

with

$$A = \begin{pmatrix} 0_{n-1\times 1} & I_{n-1\times n-1} \\ 0 & 0_{1\times n-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0_{n-1\times 1} \\ 1 \end{pmatrix}.$$

This paper deals with the exponential stabilization of chained form system by means of a static discontinuous feedback. It is divided into two parts; the first one is devoted to the exponential stabilization for all initial conditions such that $x_0(0) \neq 0$; this result is then extended to a global exponential stabilization in Section 3.

The notations used in this paper are standard. For a scalar s, $|s|$ denotes its absolute value, for a vector v, $\|v\|$ denotes its euclidian norm and v_i its ith coordinate. For a square

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matrix M, $\|M\|$ and $\lambda_{\max}^M$ denote its maximum eigenvalue and $\lambda_{\min}^M$ its minimum eigenvalue. Finally, the real function sign used in the sequel is defined by $\text{sign}(r)=1$ if $r \geqslant 1$ and $\text{sign}(r)=-1$ if $r<0$.

2. Exponential stability for all $x_0(0) \neq 0$

In this section, a state feedback that almost exponentially stabilizes system (1–2) is designed. By *almost exponential stability* is meant exponential stability for any initial condition in an open dense subset of $\mathbb{R}^n$ (Astolfi, 1996). This type of stability is often encountered in this research field (see e.g. Astolfi, 1996; Canudas de Witt & Khennouf, 1995; Reyhanoglu, Cho, McClamroch, & Kolmanovsky, 1998; Jiang, 2000) and is a revealing symptom of the absence of smooth stabilizing feedback. The open dense subset characterizes in some sense the regions where the linearized system is controllable. Most of the existing work is based on a discontinuous coordinate transformation issued from the notion of σ process and, in that case, the subset turns out to be the set where the transformation is continuous.

As in Astolfi (1996), our approach is based on linear tools. However, instead of applying a coordinate change that transforms (2) into a linear system when $x_0 \neq 0$, it is proposed to consider u_0 as a time function and hence subsystem (2) as a linear time-varying system with u_1 as control. An unfortunate reflex would be to use spectrum assignment methods to place the poles of subsystem (2). Indeed, it is known that contrary to time-invariant systems, the stability of the time varying systems cannot be ensured that way (Vidyasagar, 1993; Khalil, 1996). This does not happen if the influence of the time slowly modifies the general behaviour of the system. These systems are referred as slowly time-varying systems in the literature (Vidyasagar, 1993; Khalil, 1996). The solution chosen here consists in taking u_0 slowly varying with respect to x. With this glance, one can prove that, up to a change of coordinates depending upon the control u_0 and hence upon the time, subsystem (2) is equivalent to a system that can be bounded by a time-invariant controllable system. This change of coordinates puts subsystem (2) into a controllable Brunovsky-like canonical form and the transformation proposed by Astolfi (1996) proves to be a particular case of this coordinate change (see Remark 2). This renders the design of a stabilizing state feedback for subsystem (2) easy. Moreover, it can be extended to the whole system giving an exponential converging rate to the controlled system, which is an important performance characteristic for practical applications.

The sequel is organized as follows: after assumptions insuring a slow evolution of $u_0(t)$, a transformation putting subsystem (2) into a Brunovsky-like canonical form is given. Based on this, a class of state feedbacks that stabilize subsystem (2) when u_0 slowly varies is proposed.

2.1. Preliminaries

Assumption 1. *Assume that $u_0 : \mathbb{R}^+ \to \mathbb{R}$ is a continuous almost everywhere differentiable function with*:

(1) *for all $t \geqslant 0$, $u_0(t) \neq 0$,*
(2) *for almost all $t \geqslant 0$, $\|(\mathrm{d}u_0/\mathrm{d}t)(t)\| \leqslant \alpha |u_0(t)|$, α being some real positive constant.*

If u_0 vanishes, x clearly becomes uncontrollable; first item avoids this loss of controllability. Second item only excludes controls that vanish "almost" infinitely fast.

Keeping in mind that u_0 is "almost" constant (α being chosen small enough) and never vanishes, it becomes natural to transform system (2) into its Brunovsky normal form by defining:

$$\xi(t) := T(u_0(t))x(t),$$

with

$$T(u_0(t)) := \text{diag}(1, u_0(t), \ldots, u_0(t)^{n-1}), \tag{3}$$

where ξ is solution of ordinary differential equation $\dot{\xi} = \dot{T}T^{-1}\xi + Tu_0AT^{-1}\xi + TBu_1$. Since $Tu_0AT^{-1} = A$ and $\dot{T} = (\dot{u}_0/u_0)LT$ with $L = \text{diag}(0, 1, \ldots, n-1)$, this gives

$$\dot{\xi} = \left(A + \frac{\dot{u}_0}{u_0}L\right)\xi + Bu_0^{n-1}u_1. \tag{4}$$

Remark 2. In Astolfi (1996), it is proposed to apply the transformation $\zeta_i = x_i / x_0^{i-1}$ for $i \in [1, \ldots, n]$. This transformation, also denoted state scaling, is equivalent to the coordinate change (3) with $u_0 = -kx_0$ that gives $\xi_i = x_i/(-kx_0)^{i-1}$. Hence, one has $\zeta_i = (-k)^{i-1}\xi_i$; to a certain extent, the transformation proposed by Astolfi (1996) is a particular case of coordinate change (3).

The above transformation gives the exact Brunovsky normal form if u_0 is kept constant. Eq. (4) can also been seen as a nonlinear system with a linearly bounded nonlinear term (because of Assumption 1.2). This last point makes system (4) easily stabilizable by means of a smooth state feedback as in Theorem 3. The asymptotic stability of $x(t) = T^{-1}(u_0(t))\xi(t)$ is ensured if ξ converges sufficiently fast. These last points are exposed in the forthcoming subsection.

2.2. Class of stabilizing feedbacks for subsystem (2)

Theorem 3. *Assume that $u_0(t)$ fulfills Assumption 1, then $u_1(x, u_0(t)) := -u_0^{1-n}(t)B^{\mathrm{T}}PT(u_0(t))x$, with P being the symmetric definite positive solution of the following Riccati equation parameterized by some real positive constant $\gamma > (n-1)\alpha$, is well defined and exponentially stabilizes subsystem (2) to the origin.*

$$-2PBB^{\mathrm{T}}P + A^{\mathrm{T}}P + PA + (2\gamma + \alpha)P + \alpha LPL = 0. \tag{5}$$

Theorem 3 is proved in Appendix A. Note that, thanks to Assumption 1, u_0 never vanishes. Eq. (5) arises in a variety of stochastic control problems. The pair (A,B) being controllable, Eq. (5) is known to admit a maximal solution P such that $A-BB^{\mathrm{T}}P$ has all its eigenvalues in the open left half plane (Wonham, 1970). Hence, u_1 is properly defined but nothing a priori ensures that it remains bounded since this depends upon the choice of $u_0(t)$. Note that P is *static and independent of* u_0.

As a particular case of Theorem 3, one has

Theorem 4. *The feedback*

$$u_0(x_0,x) = -kx_0, \tag{6}$$

$$u_1(x_0,x) = -(-kx_0)^{1-n}B^{\mathrm{T}}PT(-kx_0)x, \tag{7}$$

exponentially stabilizes system (1–2) *for any initial condition in the open dense set* $\{(x_0,x)\in\mathbb{R}^{n+1}\,|\,x_0\neq 0\}$ *as soon as* $\gamma>(n-1)k$ *and* $k>0$. *Furthermore, along the trajectories of the closed-loop system, the feedback law* (6–7) *is well defined, bounded for all* $t\geqslant 0$ *and exponentially tends to zero*.

Note that x_0 satisfies the differential equation $\dot{x}_0=-kx_0$; therefore, if $x_0(0)\neq 0$, for all $t\geqslant 0$ $x_0(t)\neq 0$ and the singularity set $\{(x_0,x)\in\mathbb{R}^{n+1},\ x_0=0\}$ is never crossed: for any initial condition outside this set, there is always a unique and smooth solution to the closed-loop system. Note also that u_1 becomes larger when $x_0(0)$ gets closer to the origin. This drawback underlined, among others, by Luo and Tsiotras (2000) is recurrent to almost stabilization.

3. Global exponential stability

In the above section, no solution is brought to the stabilization problem for an initial condition on the submanifold $x_0(0)=0$. The purpose of this section is to answer this question by proposing a globally exponentially stabilizing static state feedback.

The following definition of global exponential stability can be found in most books related to stability. This notion is sometimes called K-exponential stability to emphasize that function $h(\cdot)$ has not to be linear.

Definition 5. Consider the nonlinear time-invariant system $\Sigma:\dot{x}=f(x)$ with $x\in\mathbb{R}^n$. Let $h:\mathbb{R}^+\to\mathbb{R}^+$ be of class K if it is continuous, monotonically increasing and $h(0)=0$. System Σ is said globally exponentially stable iff there exists a strictly positive constant λ and a function h of class K such that $\forall x(0)\in\mathbb{R}^n$, $\forall t\geqslant 0$, $\|x(t)\|\leqslant h(\|x(0)\|)\mathrm{e}^{-\lambda t}$.

Extending almost global stability of Section 2 to global exponential stability necessarily implies to take into account the singularity $x_0(0)=0$. This step is often bypass or shortly addressed. Most of the time, it is proposed to first apply an open-loop control during some a priori fixed time t_s in order to steer the state away from the singularity and then to switch to the exponentially Converging feedback (see Astolfi, 1996; Canudas de Witt & Khennouf, 1995; Reyhanoglu et al., 1998). An improvement has been brought in Jiang (2000) who proposed to take u_0 constant and u_1 as a backstepping based control that makes x converge to the origin. Unfortunately, until today, nothing better than *global exponential regulation* was obtained. Exponential regulation is defined as exponential stability except that h is not required to be of class K. Both ensure an exponential converging rate but, contrary to exponential stability, with exponential regulation, transient deviations cannot be over-bounded with a bound vanishing with the initial condition. Hence, one loses the highly desirable property "the more closer to the origin the trajectory starts, the closer to the origin it remains". Such a bound on the closed-loop trajectory could only be obtained using time-varying tools. Unfortunately, as pointed out by Murray (1991) and Gurvits (1992), smooth periodic time-varying feedback can only yield algebraic converging rate; global exponential stability requires nonsmooth (see Sørdalen & Egeland, 1995; M'Closkey & Murray, 1997; Godhavn & Egeland, 1997) or aperiodic (see Yu & Shihua, 2000) time-varying state feedback.

The following theorem gives (to the authors knowledge) the first globally exponentially stabilizing static state feedback for chained form systems. It is obtained by switching between two smooth static feedbacks, the key point being the choice of the switching surface. Note that contrary to preceding works on this subject, the switch only depends upon the states and not upon the time, resulting in a static discontinuous feedback law in the classical sense and not a "*time-dependent static feedback*". Moreover, this feedback is bounded for bounded states.

3.1. Main result

Theorem 6. *Let*

- β, δ, k *and* γ *be strictly positive real constants such that* $\gamma>(n-1)k>0$,
- P *be the solution of static Riccati equation* (5),
- V *be defined by* $V(x,u_0):=x^{\mathrm{T}}T(u_0)PT(u_0)x$ *with* $T(u_0):=\mathrm{diag}(1,\ldots,u_0^{n-1})$,
- $\Gamma:=\{(x_0,x)\in\mathbb{R}\times\mathbb{R}^n,\ s.t.\ V(x,-kx_0)<\delta[kx_0]^{2\varepsilon}\}\cup\{(0,0)\}$ *with* ε *being a real constant such that* $\gamma/k>\varepsilon>n-1$,

Then, the static discontinuous feedback

$$u_0(x_0,x)=\begin{cases}\mathrm{sign}(x_0)\beta & \text{if } (x_0,x)\notin\Gamma,\\ -kx_0 & \text{if } (x_0,x)\in\Gamma,\end{cases} \tag{8}$$

$$u_1(x_0,x)=\begin{cases}0 & \text{if } u_0=0,\\ -u_0^{1-n}B^{\mathrm{T}}PT(u_0)x & \text{otherwise},\end{cases} \tag{9}$$

globally exponentially stabilizes chained form system (1–2). *Moreover, the feedback law is bounded over bounded subsets of* $\mathbb{R}\times\mathbb{R}^n$.

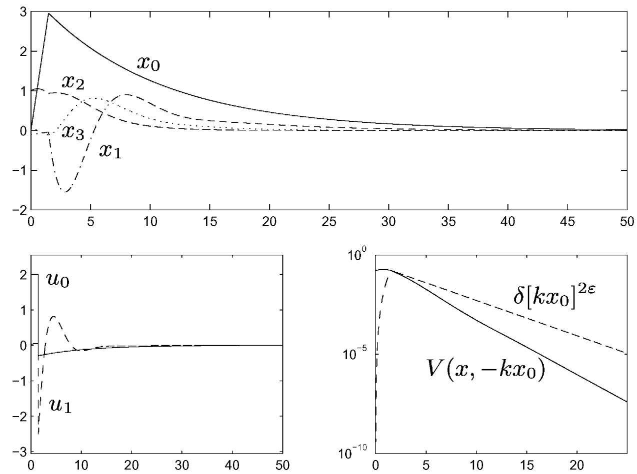

Fig. 1. Behavior of a chained system of dimension four with feedback (8–9), initial condition $[0, 1, 0.1, -0.1]$ (same as in Sørdalen and Egeland (1995)) and parameters $[k, \beta, \gamma, \delta, \varepsilon] = [0.1, 2, 0.21, 25, 2.05]$. Note the evolutions of $V(x, -kx_0)$ and $\delta[kx_0]^{2\varepsilon}$ emphasizing that $\delta[kx_0]^{2\varepsilon} > V(x, -kx_0)$ for all $t > t_s \approx 1.5$ s. Note that after t_s, feedbacks (8–9) and (6–7) are identical.

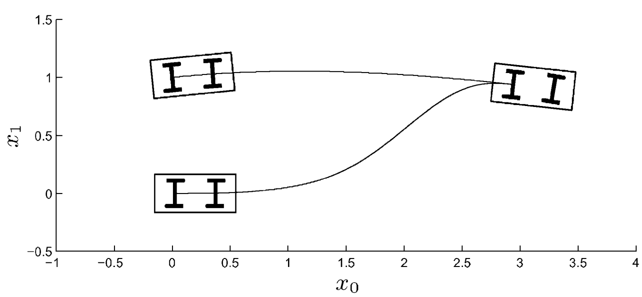

Fig. 2. The resulting path in the xy-plane. The variables $x = x_0$ and $y = x_1$ are interpreted as the planar position of a four-wheeled car.

Theorem 6 is proved in Appendix B and Figs. 1 and 2 show the behaviour of system (1–2) with feedback (8–9). The key feature of the above feedback is that the constant control $\operatorname{sign}(x_0)\beta$ is applied only as long as x_0 is too small with respect to x in order to retrieve "*some sufficient*" controllability on the state x. Note that along the trajectories of the closed-loop system, when u_0 tends to zero, u_1 does the same (see Appendix B). Hence, "$u_1 = 0$ *if* $u_0 = 0$" is, on the trajectories of the controlled system, the continuous extension of "$u_1 = -u_0^{1-n}B^{\mathrm{T}}PT(u_0)x$ *otherwise*" and $u_0 = 0$ is applied iff the system starts at the origin. It should be emphasized that a discontinuous feedback may become excessively large even for small states. In particular, it may happens for initial conditions close to a singular manifold as in previous works on this field (see Luo & Tsiotras, 2000, and the references therein). Here, the feedback law being bounded over bounded subsets of $\mathbb{R} \times \mathbb{R}^n$, this phenomena is avoided.

4. Conclusion

In this paper a novel transformation for single chained form system was proposed. This enables to derive a class of almost exponential stabilizing feedbacks (u_0, u_1), provided that u_0 varies sufficiently slow. As a particular case, one retrieves a previously existing result. This feedback was extended to a global exponential stabilization. The so obtained static discontinuous feedback law has the advantageous property of being bounded for bounded states and converging to zero along the trajectories of the closed-loop system. All the control laws proposed in this paper can be straightforwardly extended to multi-input chained systems.

Appendix A. Proof of Theorems 3 and 4

A.1. *x is exponentially stable if $\gamma > (n-1)\alpha$*

Note first that $\xi(t)$ exists for all initial conditions. Taking $V(\xi) = \xi^{\mathrm{T}}P\xi$, it directly follows, thanks to Riccati equation (5) and Assumption 1, that $\dot{V} \leqslant -2\gamma V$. Since V is continuous, ξ is exponentially stable. Now, using Assumption 1 and $\xi = T(u_0)x$, it follows with $k(u_0) := \max(|u_0(0)|, |u_0(0)|^{-1})^{n-1}$ that:

$$\|x(t)\| \leqslant \|x(0)\|k(u_0)\sqrt{\frac{\lambda_{\max}^P}{\lambda_{\min}^P}}\mathrm{e}^{((n-1)\alpha-\gamma)t}.$$

A.2. *Feedback law (6–7) is exponentially bounded*

Recalling that along the trajectories of the closed-loop system one has $V(t) = x^{\mathrm{T}}TPTx \leqslant V(0)\mathrm{e}^{-2\gamma t}$, it follows $|u_1(t)| \leqslant |kx_0(0)|^{1-n}\sqrt{\lambda_{\max}^P V(0)}\mathrm{e}^{((n-1)k-\gamma)t}$.

Appendix B. Proof of Theorem 6

The proof will hold in three steps. First, we shall prove that any trajectory starting from Γ remains in Γ implying with Theorem 4 that the trajectory exponentially converges to the origin. In a second step, this exponential regulation will be proved to be exponential stabilization in the usual sense. Finally, the control will be shown to be bounded over bounded subsets of $\mathbb{R} \times \mathbb{R}^n$.

First, let us recall two important properties used in the sequel. Let u_0 fulfilling Assumption 1 and use $(u_0, -u_0^{1-n}(t)B^{\mathrm{T}}PT(u_0(t))x)$ to control system (1–2), then, for all $\forall t \geqslant t_0$:

$$V(x(t), u_0(t)) \leqslant V(x(0), u_0(0))\mathrm{e}^{-2\gamma(t-t_0)}, \tag{B.1}$$

where t_0 denotes the initial instant. Note also that, by definition of V, for all $(x_0, x) \in \mathbb{R} \times \mathbb{R}^n$, one has

$$V(x, -kx_0) \leqslant \frac{\max(1, k|x_0|)^{2n-2}}{\min(1, \beta)^{2n-2}} V(x, \operatorname{sign}(x_0)\beta). \tag{B.2}$$

B.1. Exponential regulation of system (1–2)

Let $(x_0(0), x(0)) \notin \Gamma$. It then follows that $u_0 = \text{sign}(x_0)\beta$ and hence that $|x_0(t)| = |x_0(0)| + \beta t$ is strictly increasing and that u_0 satisfies Assumption 1 with $\alpha = 0$ [1]. It ensures that $V(x, \text{sign}(x_0)\beta)$ is a continuous and exponentially decreasing time function along the trajectories of the closed-loop system. Using Eq. (B.2), it follows that $V(x, -kx_0)$ necessarily becomes lower than $\delta[kx_0]^{2\varepsilon}$ after some time. Therefore, for any initial condition outside Γ, there necessarily exists some time t_s such that (x_0, x) reaches Γ.

Let t_s is an instant such that $(x_0(t_s), x(t_s)) \in \Gamma$. If $(x_0(t_s), x(t_s)) = (0,0)$, the applied control being (0,0), the trajectory clearly remains in Γ. If now $(x_0(t_s), x(t_s)) \neq (0,0)$ then $\forall t \geqslant t_s$, $|x_0(t)|^{2\varepsilon} = |x_0(t_s)|^{2\varepsilon} \mathrm{e}^{-2k\varepsilon(t-t_s)}$ and using Eq. (B.1) with $t_0 = t_s$ and $\gamma > k\varepsilon$, it follows that for all $t \geqslant t_s$, $(x_0(t), x(t))$ remains in Γ. Hence, *Γ is an invariant subset of $\mathbb{R} \times \mathbb{R}^n$ for system* (1–2) under control (8–9).

The discontinuity of the feedback is only reached once: the feedback law is *piecewise smooth*. Moreover on Γ, feedbacks (8–9) and (6–7) are identical; so any trajectory reaching Γ at some time t_s exponentially converges to the origin. This concludes the proof of this subsection. To obtain the exponential stability, it is proved in the next section that the transitory excursions of the closed-loop trajectories are bounded by some upper-bound that goes to zero as the initial condition goes closer to the origin (recall the definitions mentioned above).

B.2. Exponential stabilization of system (1–2)

Let $t_s(x_0(0), x(0))$ be now the first time instant such that $V(x(t_s), -kx_0(t_s)) \leqslant \delta[kx_0(t_s)]^{2\varepsilon}$. Then for all $t < t_s$ (if there is some), the applied control u_0 is $\text{sign}(x_0)\beta$ while u_0 switches to $-kx_0$ after t_s.

B.2.1. t_s is bounded by some $\bar{t}_s(x_0(0), x(0))$

From Eqs. (B.2) and (B.1), it follows:

$$V(x(t_s), -kx_0(t_s)) \leqslant \frac{\max(1, k|x_0(t_s)|)^{2n-2}}{\min(1,\beta)^{2n-2}} \times V(x(0), \text{sign}(x_0(0))\beta)\mathrm{e}^{-2\gamma t_s}.$$

Since $V(x,u) = x^{\mathrm{T}} T(u) P T(u) x$, it gives

$$V(x(t_s), -kx_0(t_s)) \leqslant \max(1, k|x_0(t_s)|)^{2\varepsilon} \times \max(\beta^{-1}, \beta)^{2n-2} \|x(0)\|^2 \lambda_{\max}^P \mathrm{e}^{-2\gamma t_s}.$$

Since t_s is the first instant such that $V(x(t_s), -kx_0(t_s)) \leqslant \delta[kx_0(t_s)]^{2\varepsilon}$ and since $|x_0(t_s)| = |x_0(0)| + \beta t_s$, it follows:

$$\delta \leqslant \max(1, k|x_0(0)| + k\beta t_s)^{-2\varepsilon} \max(\beta^{-1}, \beta)^{2n-2} \times \|x(0)\|^2 \lambda_{\max}^P \mathrm{e}^{-2\gamma t_s}.$$

This gives $t_s \leqslant \bar{t}_s(x_0(0), x(0)) := \max(0, \tau_1, \tau_2)$ with

$$\tau_1 := \frac{1}{2\gamma} \log\left[\frac{1}{\delta} \|x(0)\|^2 \lambda_{\max}^P \max(\beta^{-1}, \beta)^{2n-2}\right],$$

$$\tau_2 := \frac{k}{\beta}\left[\sqrt[2\varepsilon]{\frac{\|x(0)\|^2}{\delta} \lambda_{\max}^P \max(\beta^{-1}, \beta)^{2n-2}} - \frac{|x_0|}{k}\right],$$

where τ_1 and τ_2 being, respectively, the roots of $\delta = \max(\beta^{-1}, \beta)^{2n-2} \|x(0)\|^2 \lambda_{\max}^P \mathrm{e}^{-2\gamma t}$ and $\delta = [k|x_0(0)| + k\beta t]^{-2\varepsilon} \max(\beta^{-1}, \beta)^{2n-2} \|x(0)\|^2 \lambda_{\max}^P$. $\bar{t}_s$ is forced to be positive with the choice $\max(0, \tau_1, \tau_2)$. With this constraint, $\bar{t}_s$ is well defined and *continuous* on all $\mathbb{R} \times \mathbb{R}^n$.

B.2.2. x_0 is bounded by some $\rho_0(x_0(0), x(0))$

For all $t < t_s$ (if $t_s > 0$), x_0 is given by $|x_0(t)| = |x_0(0)| + \beta t$ and by $|x_0(t)| = |x_0(t_s)|\mathrm{e}^{-k(t-t_s)}$ for all $t \geqslant t_s$. Hence, for any initial condition, using $0 \leqslant t_s \leqslant \bar{t}_s(x_0(0), x(0))$, it follows:

$$\forall t \geqslant 0, \quad |x_0(t)| \leqslant \rho_0(x_0(0), x(0))\mathrm{e}^{-kt},$$

$$\rho_0 := [|x_0(0)| + \beta \bar{t}_s(x_0(0), x(0))]\mathrm{e}^{k\bar{t}_s(x_0(0), x(0))}. \tag{B.4}$$

Note that ρ_0 is *continuous* with respect to its arguments since so is $\bar{t}_s(x_0(0), x(0))$ and vanishes as the initial condition goes to zero.

B.2.3. x is bounded by some $\rho(x_0(0), x(0))$

For all $t < t_s$ (if $t_s > 0$) the exponential decrease of $V(x, \text{sign}(x_0)\beta)$ guarantees that:

$$\forall t < t_s, \quad \|x(t)\| \leqslant r_1(x_0(0), x(0))\mathrm{e}^{-\gamma t},$$

$$\text{with}: \quad r_1 := \sqrt{\frac{\lambda_{\max}^P}{\lambda_{\min}^P}} \max(\beta^{-1}, \beta)^{n-1} \|x(0)\|. \tag{B.5}$$

For all $t \geqslant t_s$, $V(x, -kx_0)$ is exponentially decreasing: $V(x(t), -kx_0(t)) \leqslant V(x(t_s), -kx_0(t_s))\mathrm{e}^{-2\gamma(t-t_s)} \leqslant \delta[kx_0(t_s)]^{2\varepsilon} \mathrm{e}^{-2\gamma(t-t_s)}$. By definition of V: $\|x(t)\|^2 \lambda_{\min}^P \min(1, k|x_0(t)|)^{2n-2} \leqslant V(x(t), -kx_0(t))$. These two last equations give

$$\|x(t)\|^2 \leqslant \frac{\delta}{\lambda_{\min}^P} \max(1, k|x_0(t_s)|)^{2n-2} \times [kx_0(t_s)]^{2(\varepsilon-n+1)} \mathrm{e}^{-2[\gamma - k(n-1)](t-t_s)}.$$

Finally, using t_s's bound and $\varepsilon > n-1$, one gets

$$\forall t \geqslant t_s, \quad \|x(t)\| \leqslant r_2(x_0(0), x(0))\mathrm{e}^{-[\gamma - k(n-1)]t}$$

$$\text{with}: \quad r_2 := \frac{\delta^{1/2}}{(\lambda_{\min}^P)^{1/2}} \max(1, k|x_0(0)| + k\beta \bar{t}_s)^{n-1} \times [k|x_0(0)| + k\beta \bar{t}_s]^{\varepsilon - n + 1} \mathrm{e}^{[\gamma - k(n-1)]\bar{t}_s}. \tag{B.6}$$

Eq. (B.5) coupled with (B.6) gives

$$\forall t \geqslant 0, \quad \|x(t)\| \leqslant \rho(x_0(0), x(0))\mathrm{e}^{-[\gamma - k(n-1)]t},$$

$$\rho := \max(r_1(x_0(0), x(0)), r_2(x_0(0), x(0))). \tag{B.7}$$

[1] Note that $x_0(t)$ remains continuously differentiable along the trajectories of system (1) with control $u_0 = \text{sign}(x_0)\beta$.

Here again, ρ is *continuous* with respect to its arguments with $\rho(0,0)=0$.

Eqs. (B.4) and (B.7) clearly end the proof: static discontinuous feedback law (8–9) globally exponentially stabilize chained form system (1–2) to the origin.

B.3. The feedback law is bounded

From Eqs. (8) and (9), it follows that before t_s, $u_0 = \mathrm{sign}(x_0)\beta$ and after t_s, $u_0(t) = -kx_0$. Hence, $u_0(x_0, x)$ is clearly bounded over bounded subset of $\mathbb{R} \times \mathbb{R}^n$. Furthermore, since x_0 is exponentially converging (recall Eq. (B.4)), $u_0(t)$ is exponentially converging to the origin.

Let us now prove the same for u_1. For $(x_0, x) \notin \Gamma$, $u_0 = \mathrm{sign}(x_0)\beta$ and $|u_1| \leqslant \beta^{1-n}\lambda^P_{\max}\max(1, \beta^{n-1})\|x\|$. Now, if $(x_0, x) \in \Gamma$, $u_0 = -kx_0$ and $V(x, -kx_0) < \delta[kx_0]^{2\varepsilon}$ giving: $|u_1| \leqslant |kx_0|^{\varepsilon+1-n}\sqrt{\delta\lambda^P_{\max}}$. Thus, since $\varepsilon > n-1$, it follows from the two last equations that $u_1(x_0, x)$ is bounded over bounded subset of $\mathbb{R} \times \mathbb{R}^n$. Furthermore, since x_0 and x exponentially converge to the origin, u_1 as time function does the same. Note also that $u_1(x_0, x)$ tends to zero as (x_0, x) tends to the origin.

References

Astolfi, A. (1996). Discontinuous control of nonholonomic systems. *Systems & Control Letters*, *27*, 37–45.

Brockett, R. W., Millman, R. S., & Sussmann, H. S. (1983). Asymptotic stability and feedback stabilization. In R. W. Brockett, R. S. Millman & H. S. Sussmann (Eds.) *Differential geometric control theory*. Birkhäuser: Boston-Basel-Stuttgart.

Canudas de Witt, C., & Khennouf, H. (1995), Quasi-continuous stabilizing controllers for nonholonomic systems: Design and robustness considerations. In: *Proceedings of the European Control Conference*, Rome, Italy, Vol. 3, 2630–2635.

Canudas de Witt, C., Siciliano, B., & Bastin, G. (Eds.) (1996). *Theory of robot control*. London: Springer.

Coron, J. M. (1992). Global asymptotic stabilization for controllable systems without drift. *Mathematics of Control, Signal and Systems*, *5*(3), 295–312.

Coron, J. M. (1995). On the stabilization in finite time of locally controllable systems by means of continuous time-varying feedback law. *SIAM Journal on Control and Optimization*, *33*(3), 804–833.

Godhavn, J. M., & Egeland, O. (1997). A Lyapunov approach to exponential stabilization of nonholonomic systems in power form. *IEEE Transactions on Automatic Control*, *42*(7), 1029–1032.

Gurvits, L. (1992). Averaging approach to nonholonomic motion planning. In *Proceedings of the IEEE international conference on robotics and automation*, pp. 2541–2546.

Isidori, A. (1995). *Nonlinear control theory*. Berlin-Heidelberg: Springer.

Jiang, Z. P. (2000). Robust exponential regulation of nonholonomic systems with uncertainties. *Automatica*, *36*(2), 2000.

Khalil, H. K. (1996). *Nonlinear systems*. London: Prentice-Hall.

Kolmanovky, I., & McClamroch, N. H. (1995). Developments in nonholonomic control problems. *IEEE Control Systems Magazine*, *15*(6), 20–36.

Lin, W. (1996). Time-varying feedback control of nonaffine nonlinear systems without drift. *Systems & Control Letters*, *29*(2), 101–110.

Luo, J., & Tsiotras, P. (2000). Control design for chained-form systems with bounded inputs. *Systems & Control Letters*, *39*(2), 123–131.

M'Closkey, R. T., & Murray, R. M. (1997). Exponential stabilisation of driftless nonlinear control systems using homogeneous feedback. *IEEE Transactions on Automatic Control*, *42*(5), 614–628.

Morin, P., & Samson, C. (1997). Application of backstepping techniques to the time-varying exponential stabilization of chained form systems. *European Journal of Control*, *3*(1), 15–36.

Murray, R. M. (1991). Control of nonholonomic systems using chained form. In M. Enos (Ed.), *Dynamic and control of mechanical systems. The falling cat and related problems*, Vol 1 (pp 219–245). The Field Institute for Research in Mathematical Sciences.

Murray, R. M., & Sastry, S. (1993). Nonholonomic motion planning: Steering with sinusoids. *IEEE Transactions on Automatic Control*, *38*, 700–716.

Reyhanoglu, M., Cho, S., McClamroch, N. H., & Kolmanovsky, I. (1998). Discontinuous feedback control a planar rigid body with an underactuated degree of freedom. In *Proceedings of 37th conference on decision and control*, Piscataway, NJ, USA, pp. 433–438.

Samson, C. (1995). Control of chained systems, application to path following and time-varying point stabilization of mobile robot. *IEEE Transactions on Automatic Control*, *40*(1), 64–77.

Sørdalen, O. J., & Egeland, O. (1995). Exponential stabilization of nonholonomic chained systems. *IEEE Transactions on Automatic Control*, *40*(1), 35–49.

Teel, A. R., Murray, R. M., & Walsh, G. C. (1995). Non-holonomic control systems: From steering to stabilization with sinusoids. *International Journal of Control*, *62*(4), 849–887.

Vidyasagar, M. (1993). *Nonlinear systems analysis* (2nd ed.). Englewood Cliffs, NJ: Prentice-Hall.

Wonham, W. M. (1970). *Probabilistic methods in applied mathematics*, Vol. 2, Random differential equations in control theory (pp. 131–212). New York: Academic Press.

Yu, P. T., Shihua, L. (2000). Smooth exponential stabilization of nonholonomic systems via time-varying feedback. In *The 39th Proceedings of the IEEE Conference on Decision and Control*, Piscataway, NJ, USA, Vol. 2 (pp. 1912–1917).

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