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Constrained Minimum-Time-Oriented Feedback Control for the Stabilization of Nonholonomic Systems in Chained Form

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Abstract. In this paper, a discontinuous state-feedback law is proposed for the stabilization of nonholonomic systems in power form. The feedback law is based on a receding-horizon strategy in which the open-loop optimization problem is a minimum-time steering process. Suboptimal formulations are used explicitly to meet the real-time implementability requirements. Stability is established in a sampled-data context and illustrative simulations are given to show the effectiveness and the real-time implementability of the proposed scheme.

Key Words. Nonholonomic systems, sampled control, saturated control, stabilization, receding-horizon control.

1. Introduction

Consider a nonholonomic system in the power form proposed in Ref. 1 together with saturation constraints on the control inputs,

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_{i+1} = x_i u_1, \qquad i \in \{2, \ldots, n-1\}, \tag{1}$$

$$-u_j^{\max} \leq u_j \leq u_j^{\max}, \qquad j \in \{1, 2\}. \tag{2}$$

This system and more generally nonholonomic systems have attracted much attention over the past few years within the nonlinear control community. The most interesting feature that may explain this attention is the fact that nonholonomic systems do not satisfy the Brockett conditions (Ref. 2). These

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are necessary conditions for the existence of time-invariant C^1-static stabilizing feedbacks. The absence of feedback linearizing transformations (Ref. 3) makes these systems difficult to stabilize.

A general discussion on the stabilization of nonholonomic systems has been proposed in Ref. 4. Local small-time stabilizability by continuous time-varying feedbacks has been proved in Ref. 5, initiating a wide literature on time-varying asymptotic stabilization; see Refs. 5–11.

Local asymptotic stabilization by time-invariant discontinuous feedback laws has been proposed in Ref. 12, while a minimum-energy based discontinuous receding-horizon control has been applied for global asymptotic stabilization in Refs. 13–14. A discontinuous approach has been proposed also in Ref. 15, where the problem of high inputs that arises generally in the neighborhood of the origin has been handled by monitoring the controller parameters. Another approach based on state space subdivision into good/bad regions has been proposed in Ref. 16. Finite-time stabilization by use of switching hybrid controllers can be found in Ref. 17.

Controllers for asymptotic tracking of reference trajectories have been designed using backstepping tools (Refs. 18–21) or a flatness-based approach (Ref. 22). Despite the extremely rich literature on the subject and to the best of our knowledge, the problem of minimum-time stabilization of (1) under the saturation constraint (2) has never been addressed in the general n-dimensional case.

The key result of the present paper is to provide a sampled data, receding-horizon feedback strategy that provides the practical stabilization of (1) around some desired state. Furthermore, the proposed feedback is oriented toward minimum-time steering. More precisely, at each sample time, a trajectory from the current state to the desired state is produced by exploiting a constructive controllability property of (1). Moreover, the simple nature of this controllability construction allows one to find easily trajectories that meet the saturation constraint (2). Finally, using the driftless nature of the system, one may scale locally the inputs and their durations without changing the points that are visited in the state space. This allows one to use potentially more of the available input at each step to transfer the system from the current state to the desired one in shorter time.

This trajectory generation algorithm is then used in a receding-horizon fashion to produce a sampled data feedback law. The overall methodology provides an effective approach for practical stabilization. Such stabilization is of interest for driftless systems since every state can be an equilibrium point. In particular, whenever the controls are turned off, the system stops dead.

In this paper, $\delta > 0$ denotes some fixed sampling period. For all $v \in \mathbb{R}^i$, the supremum norm is defined by

$$\|v\|_\infty := \max_{1 \le j \le i} |v_j|.$$

Given any integer $k \in \mathbb{N}$, k^+ stands for $k+1$. For any time-dependent signal $v(\cdot)$, $v(k)$ simply denotes $v(k\delta)$. When the state $x \in \mathbb{R}^n$ is clearly known from the context, z is used to designate the substate $(x_2, \dots, x_n)^T \in \mathbb{R}^{n-1}$. The classical notation $X(t; 0; x^0; u)$ is used to denote the solution at instant t of (1) starting from the initial condition $(0, x^0)$ under the control $u(\cdot)$. For any positive real r, $\bar{E}(r)$ denotes the nearest bigger integer, namely

$$\bar{E}(r) := \min_{k \in \mathbb{N}} \{k \ge r\}.$$

Finally

$$U_{\text{ad}} := [-u_1^{\max}, u_1^{\max}] \times [-u_2^{\max}, u_2^{\max}]$$

denotes the set of admissible controls.

The paper is organized as follows. In Section 2, the open-loop optimization problem that is used to define the receding-horizon feedback control is presented together with a fast algorithm yielding suboptimal solutions. The corresponding receding-horizon feed-back is defined in Section 3 where stability results are also established. In Section 4, the paper ends with some numerical simulations illustrating the real-time implementability of the proposed control law and showing its effectiveness in some typical situations including initial conditions that are classically viewed as singular.

2. Open-Loop Optimization Problem

The feedback law proposed in this paper is based on the receding-horizon principle. Recall that such a feedback is obtained by solving at each sampling instant $k\delta$ an open-loop optimization problem in which the current state $x(k)$ plays the role of initial state. Then, the first part of the optimal control sequence is applied. At the next sampling instant k^+, a new open-loop optimization problem is solved with $x(k^+)$ as initial state, the first part of the resulting optimal sequence is applied, and so on.

Therefore, a crucial issue in the definition of a receding-horizon control scheme is the parametrization of the open-loop control profile over the future prediction horizon. In this paper, this parametrization is defined by a strictly increasing sequence of n future instants, namely,

$$\tilde{t} := (\tilde{t}_1, \tilde{t}_2, \dots, \tilde{t}_n) \in \mathbb{R}^n, \tag{3}$$

and by the sequence of corresponding controls,

$$\tilde{u} := (\tilde{u}^1, \tilde{u}^2, \ldots, \tilde{u}^n) \in U^n_{\text{ad}}. \tag{4}$$

The open-loop control profile is therefore defined by

$$\forall t \in [\tilde{t}_{j-1}, \tilde{t}_j[, \qquad U(t, \tilde{t}, \tilde{u}) := \tilde{u}^j, \qquad \tilde{t}_0 = 0 \text{ by convention}, \tag{5}$$

which defines clearly a piecewise constant control over $[\tilde{t}_1, \tilde{t}_n[$.

Since minimum-time stabilization has to inspire the feedback law, it seems natural to consider the following open-loop optimization problem

$$(P(x^0)) \qquad \min_{\tilde{t}, \tilde{u}} \tilde{t}_n, \qquad \text{under } X(\tilde{t}_n; 0; x^0; U(\cdot, \tilde{t}, \tilde{u})) = x^d, \tag{6}$$

where x^d is the desired final state.

The exact solution of (6) is quite hard to perform in a real-time context. Therefore, a fast suboptimal solution needs to be developed in order for the overall receding-horizon strategy to be implementable. The fast suboptimal solution is computed in two steps:

Step 1. First, an admissible solution is computed, namely, a solution that respects the final equality constraint in (6).

Step 2. Then, the admissible solution is improved w.r.t the cost function in (6), namely, the steering time $\tilde{t}_n$.

Since there is no unique solution for the first step, minimization is done on a parametrized set of such possible solutions.

The key point in obtaining an admissible solution is to note that, when $x^0_1 \neq x^d_1$, the following parametrization is an admissible solution for (6) for all $\tau > 0$

$$\bar{t}(\tau) := (0, \tau, \ldots, (n-1)\tau), \tag{7}$$

$$\bar{u}(\tau, x^0) := \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} u^0_1(\tau, x^0) \\ v^0_1(\tau, x^0) \end{bmatrix}, \ldots, \begin{bmatrix} u^0_1(\tau, x^0) \\ v^0_{n-1}(\tau, x^0) \end{bmatrix} \right), \tag{8}$$

where

$$u^0_1(\tau, x^0) := -(x^0_1 - x^d_1)/(n-1)\tau \tag{9}$$

and $(v^0_1 \cdots v^0_{n-1})^T \in \mathbb{R}^{n-1}$ is the piecewise constant control sequence with sampling period τ that steers to $(x^d_2, \ldots, x^d_n)^T$, the state of the z-subsystem

$$\dot{z} = u^0_1 A z + B v, \tag{10}$$

that is obtained from (1) by taking $u_1 \equiv u^0_1(\tau, x^0)$ and $z := (x_2, \ldots, x_n)^T$. Note that $(v^0_1 \cdots v^0_{n-1})^T \in \mathbb{R}^{n-1}$ is well defined, since (10) is controllable for all $u^0_1 \neq 0$. While $U(\cdot, \bar{t}(\tau), \bar{u}(\tau, x^0))$ steers clearly the state x to x^d, the fact that there is a τ such that the saturation constraint is respected lies on the following result.

Lemma 2.1. For all x^0 such that $x_1^0 \neq x_1^d$, there exists a sufficiently high $\tau > 0$ such that the open-loop control law defined by (7)–(8) is admissible and steers the system state to x^d in finite time $(n-1)\tau$, namely,

$$X(\bar{t}_n(\tau); 0; x^0; U(\cdot, \bar{t}(\tau), \bar{u}(\tau, x^0))) = x^d; \qquad \bar{u}(\tau, x^0) \leq U_{\mathrm{ad}}^n. \tag{11}$$

Proof. The equality in (11) is a straightforward consequence of the very definition of $(\bar{t}(\tau), \bar{u}(\tau, x^0))$. Only the inequality in (11) is to be proved. For $j = 1$, the inequality

$$|\bar{u}_j^i(\tau, x^0)| \leq u_j^{\max}$$

is satisfied clearly for τ sufficiently high, since τ appears in the denominator of (9). As for $j = 2$, more detailed argumentation has to be developed. Let $\tau_0 > 0$ be some arbitrary real number and let us use the parametrization

$$\tau = \mu\tau_0.$$

The result would be obtained clearly if it can be shown that $\|\bar{u}_2(\mu\tau_0, x^0)\|_\infty$ tends to 0 when μ tends to infinity. This property is a direct consequence of the fact that, by definition, $\bar{u}(\tau, x^0)$ is given by

$$(\bar{u}_2^2(\tau, x^0), \ldots, \bar{u}_2^n(\tau, x^0)) \\ := [\Psi_{n-1}^\tau(\bar{u}_1(\tau, x^0))]^{-1} [-\Phi_{n-1}^\tau(\bar{u}_1(\tau, x^0))z^0 + z^d], \tag{12}$$

where the matrices $\Psi_{n-1}^\tau(\bar{u}_1(\tau, x^0))$ and $\Phi_{n-1}^\tau(\bar{u}_1(\tau, x^0))$ satisfy the following properties:

$$\Psi_{n-1}^{\mu\tau_0}(\bar{u}_1(\mu\tau_0, x^0)) = \mu\Psi_{n-1}^{\tau_0}(\bar{u}_1(\tau_0, x^0)), \tag{13}$$

$$\Phi_{n-1}^{\mu\tau_0}(\bar{u}_1(\mu\tau_0, x^0)) = \Phi_{n-1}^{\tau_0}(\bar{u}_1(\tau_0, x^0)). \tag{14}$$

Indeed, (13)–(14) together with (12) give clearly

$$\bar{u}_2(\mu\tau_0, x^0) = (1/\mu)\bar{u}_2(\tau_0, x^0), \tag{15}$$

that tends to zero when μ tends to infinity. □

In order to improve the admissible solution $(\bar{t}(\tau), \bar{u}(\tau, x^0))$, the following map is defined for all $j \in \{1, \ldots, n\}$ and all $\lambda > 0$:

$$\theta_j^\lambda : \mathbb{R}^n \times \mathbb{R}^{2\times n} \to \mathbb{R}^n \times \mathbb{R}^{2\times n}, \qquad (\tilde{t}, \tilde{u}) \rightsquigarrow (\bar{t}, \bar{u}), \tag{16}$$

such that

(i) $\quad \bar{t}_k = \tilde{t}_k, \qquad$ for all $k < j$,

(ii) $\bar{t}_j - \bar{t}_{j-1} = (1/\lambda)(\tilde{t}_j - \tilde{t}_{j-1})$,

(iii) $\bar{t}_k - \bar{t}_{k-1} = \tilde{t}_k - \tilde{t}_{k-1}$, for all $k \in \{j+1, \dots, n\}$,

(iv) $\bar{u}^k = \tilde{u}^k$, for all $k \neq j$,

(v) $\bar{u}^j = \lambda \tilde{u}^j$;

namely, $\theta_j^\lambda(\tilde{t}, \tilde{u})$ is the control parametrization obtained from $(\tilde{t}, \tilde{u})$ by scaling the time interval $[\tilde{t}_{j-1}, \tilde{t}_j]$ by $1/\lambda$ and scaling the corresponding control $\tilde{u}^j$ by λ. In what follows, such $(\tilde{t}, \tilde{u})$ and $(\bar{t}, \bar{u})$ are said to be equivalent, namely,

$$(\tilde{t}, \tilde{u}) \sim (\bar{t}, \bar{u}), \qquad \text{iff } \exists (j, \lambda) \text{ such that } (\bar{t}, \bar{u}) = \theta_j^\lambda(\tilde{t}, \tilde{u}). \tag{17}$$

The transformation (16) is used hereafter to improve the admissible solution (7)–(8). This is because equivalent parametrizations in the sense of (17) correspond to the same intermediate states trajectories. Only the corresponding time instants differ. The following lemma states precisely this invariance property.

Lemma 2.2. For all $i \in \{1, \dots, n\}$ and all $x^0 \in \mathbb{R}^n$,

$$\{(\tilde{t}, \tilde{u}) \sim (\bar{t}, \bar{u})\}$$
$$\Rightarrow \{X(\bar{t}_i; 0; x^0; U(.; \bar{t}, \bar{u})) = X(\tilde{t}_i; 0; x^0; U(.; \tilde{t}, \tilde{u}))\}. \tag{18}$$

Proof. Recall that $(\tilde{t}, \tilde{u}) \sim (\bar{t}, \bar{u})$ iff there is some integer $j \in \{1, \dots, n\}$ and some positive real $\lambda > 0$ such that

$$(\bar{t}, \bar{u}) = \theta_j^\lambda(\tilde{t}, \tilde{u}).$$

Recall that this transformation amounts to modify simultaneously the interval $[\tilde{t}_{j-1}, \tilde{t}_j]$ by the factor $1/\lambda$ and the control u applied over the new interval by the factor λ. More precisely,

$$[\bar{t}_{j-1}, \bar{t}_j] := [\tilde{t}_{j-1}, \tilde{t}_{j-1} + (1/\lambda)(\tilde{t}_j - \tilde{t}_{j-1})], \qquad \bar{u}^j = \lambda \tilde{u}^j.$$

But for a driftless system of the form

$$\dot{x} = G(x)u,$$

it is clear that, for all x^0 and all u,

$$X(t; 0; x^0; u) = X(t/\lambda; 0; x^0; \lambda u),$$

this shows clearly that, while the $\tilde{t}_i$ and $\bar{t}_i$ are different, the corresponding trajectory states are the same, namely,

$$X(\tilde{t}_i; 0; x^0, U(\cdot, \tilde{t}, \tilde{u}) = X(\bar{t}_i; 0; x^0; U(\cdot, \bar{t}, \bar{u})). \qquad \square$$

Based on the preceding facts, the following algorithm is proposed that starts with the admissible solution already discussed (Step 0) and terminates with an improved solution (Step 3) denoted by $(\hat{t}(x^0), \hat{u}(x^0))$.

Algorithm $\mathbf{A_{n-1}(x^0)}$.

Step 0. Compute the minimal $\tau \geq \delta$ s.t. $(\bar{t}(\tau), \bar{u}(\tau, x^0))$ is admissible (Lemma 2.1).
Step 1. $(\tilde{t}, \tilde{u}) \leftarrow (\bar{t}(\tau), \bar{u}(\tau, x^0))$.
Step 2. For $i = 2, n$ do the following steps:

Step 2.1. $\lambda_i^0 \leftarrow \min(u_1^{\max}/|\tilde{u}_1^i|, u_2^{\max}/|\tilde{u}_2^i|)$.
Step 2.2. $\lambda_i \leftarrow (\bar{t}_i - \bar{t}_{i-1})/k_i\delta$, where $k_i = \bar{E}((\bar{t}_i - \bar{t}_{i-1})/\lambda_i^0\delta)$.
Step 2.3. $(\tilde{t}, \tilde{u}) \leftarrow \theta_i^{\lambda_i}(\tilde{t}, \tilde{u})$.

Step 3. $(\hat{t}(x^0), \hat{u}(x^0)) \leftarrow (\tilde{t}, \tilde{u})$.

Discussion. Recall that Algorithm $A_{n-1}(x^0)$ is to be executed for x^0 satisfying $x_1^0 \neq x_1^d$. Lemma 2.1 shows then that Step 0 is feasible. Note that, in Step 0, τ is chosen greater than the sampling period δ, since τ in $(\bar{t}(\tau), \bar{u}(\tau, x^0))$ is the time between two successive decision instants and therefore cannot be lower than δ. According to Lemma 2.1, Step 1 yields $(\tilde{t}, \tilde{u})$ such that $U(\cdot, \tilde{t}, \bar{u})$ steers the state to x^d in $\tilde{t}_n(x^0)$ time units. This gives an initial admissible solution for $P(x^0)$. The aim of Step 2 is then to improve this initial solution w.r.t. the steering time criterion by applying $n-1$ successive transformations of the form $\theta_i^{\lambda_i}$. This is done by taking, at each sample time, the maximal allowable $\lambda_i \geq 1$, reducing consequently the steering time by an amount of $[(\lambda_i - 1)/\lambda_i](\tilde{t}_i - \tilde{t}_{i-1})$ without changing the steering property thanks to Lemma 2.2. Note finally that λ_i^0 is always well defined, since the condition $x_1^0 \neq x_1^d$ implies that, for all i, $\bar{u}_1^i \neq 0$.

Some properties of Algorithm $A_{n-1}(x^0)$ are discussed later; see Lemma 3.1. Let us first address the construction of suboptimal solutions for $P(x^0)$ in the general case where $x_1^0 \neq x_1^d$ does not necessarily hold.

In order for singular situations to be avoided systematically, a constant control is first applied to steer the state away from the manifold $\{x | x_1 = x_1^d\}$. This constant control is parametrized by an integer $q \in \{1, \ldots, q_{\max}\}$ and a discrete variable $\epsilon \in \{-1, 1\}$; more precisely, the following control is first applied during q sampling periods:

$$u_{q,\epsilon}(\tau) := \begin{bmatrix} \epsilon u_1^{\max} \\ 0 \end{bmatrix}, \qquad \forall \tau \in [0, q\delta], \tag{19}$$

to steer the state to

$$x^f(x^0, q, \epsilon) := X(q\delta; 0; x^0; u_{q,\epsilon}(\cdot))$$

such that

$$x_1^f(x^0, q, \epsilon) \neq x_1^d.$$

Then, Algorithm $A_{n-1}(x^f(x^0, q, \epsilon))$ is executed to compute a steering trajectory from $x^f(x^0, q, \epsilon)$ to x^d. Therefore, for a given pair $(q, \epsilon) \in \{1, \dots, q_{\max}\} \times \{-1, 1\}$, the total resulting steering time is clearly given by

$$q\delta + \hat{t}_n(x^f(x^0, q, \epsilon)),$$

where $\hat{t}(x^f(x^0, q, \epsilon))$ is delivered by the execution of Algorithm $A_{n-1}(x^f(x^0, q, \epsilon))$.

To sum up, an admissible suboptimal solution of $P(x^0)$ may be obtained by solving the following optimization problem:

$$(\hat{q}(x^0), \hat{\epsilon}(x^0)) := \operatorname{Arg} \min_{(q, \epsilon) \in \mathscr{A}(x^0)} q\delta + \hat{t}_n(x^f(x^0, q, \epsilon)), \tag{20}$$

where

$$\mathscr{A}(x^0) := \{(q, \epsilon) \in \{1, \dots, q_{\max}\} \times \{-1, 1\} | x^f(x^0, q, \epsilon) \neq x_1^d\} \tag{21}$$

and the overall retained solution of $P(x^0)$, denoted by $(t^{\text{opt}}(x^0), u^{\text{opt}}(x^0)) \in \mathbb{R}^n \times U_{\text{ad}}^n$, is then given by

$$t^{\text{opt}}(x^0) := (\delta\hat{q}(x^0), \hat{t}^2(\hat{x}^f(x^0)), \dots, \hat{t}^n(\hat{x}^f(x^0))), \tag{22}$$

$$u^{\text{opt}}(x^0) := \left(\begin{bmatrix} \hat{\epsilon}(x^0) u_1^{\max} \\ 0 \end{bmatrix}, \hat{u}^2(\hat{x}^f(x^0)), \dots, \hat{u}^n(\hat{x}^f(x^0)) \right), \tag{23}$$

where

$$\hat{x}^f(x^0) := x^f(x^0, \hat{q}(x^0), \hat{\epsilon}(x^0)). \tag{24}$$

In the following section, some properties of the open-loop solution (22)–(23) are given; then, the receding-horizon state feedback control is defined precisely. Finally, a stability result of the corresponding closed-loop system is given.

3. Feedback Definition and Stability Results

Let us start by giving some properties of Algorithm $A_{n-1}(x^0)$ given in the preceding section. This is the object of the following lemma.

Lemma 3.1. For any nonsingular state x^0, that is $x_1^0 \neq x_1^d$, Algorithm $A_{n-1}(x^0)$ gives a solution $(\hat{t}(x^0), \hat{u}(x^0))$ satisfying the following properties:

(i) For all $i \in \{2, \dots, n\}$, $\hat{t}_i(x^0) - \hat{t}_{i-1}(x^0) = k_i\delta$, for some $k_i \in \mathbb{N}$.
(ii) For all t, $d(U(t, \hat{t}(x^0), \hat{u}(x^0)), U_{\text{ad}}) = O(\delta)$.

Proof.

(i) This is a direct consequence of Step 2.2, Step 2.3, and the definition of the transformation $\theta_i^{\lambda_i}$.

(ii) Note that, when $\delta \to 0$, $\lambda_i \to \lambda_i^0 = u_j^{\max}/\bar{u}_j^i$ for some $j \in \{1, 2\}$. This with the definition of $\theta_i^{\lambda_i}$ yields

$$\hat{u}_j^i(x^0) \to \lambda_i^0 \hat{u}_j^i(x^0)|_{\text{Step 1}} = u_j^{\max}. \qquad \square$$

Note that (ii) of Lemma 3.1 gives an indication on the quality of the suboptimal solution given by Algorithm $A_{n-1}(x^0)$ w.r.t. the minimum steering time concern. Indeed, it states that, when the sampling period δ tends to 0, the resulting control approaches the boundary of the admissible domain U_{ad} and this for all t.

Let us now define precisely the receding-horizon state feedback. Let $\epsilon_d > 0$ be any desired precision and consider the following discrete-time state feedback law defined for all $\sigma \in [0, \delta[$ by the following expression:

$$u(k\delta + \sigma) = \begin{cases} U(0, t^{\text{opt}}(x(k)), u^{\text{opt}}(x(k))), & \text{if} \|x(k) - x^d\|_\infty > \epsilon_d, \\ 0, & \text{if} \|x(k) - x^d\|_\infty \le \epsilon_d. \end{cases} \tag{25}$$

In order to state the main stability result, $x_{\text{cl}}(\cdot)$ is used to denote the closed-loop system trajectories under (25). The main result is then given by the following proposition, which roughly states that, for any chosen final precision ϵ_d, the subset

$$\{x \in \mathbb{R}^n \mid \|x - x^d\|_\infty \le \epsilon_d\}$$

is globally finite-time stable w.r.t. the closed-loop dynamics given by (1) and (25), and this by using the same a priori fixed sampling period $\delta > 0$ that has not to be decreased with ϵ_d.

Proposition 3.1. Let $\delta > 0$ be fixed. For any desired final precision $\epsilon_d > 0$, there is a scalar function $V: \mathbb{R}^n \to \mathbb{R}^+$ with the following properties:

(i) $(V(x) \to 0) \Rightarrow (x \to x^d)$.

(ii) V is radially unbounded.

(iii) V is bounded over bounded subsets of $\mathbb{R}^n$.

(iv) For all $k \in \mathbb{N}$,

$$V(x_{\text{cl}}(k^+)) - V(x_{\text{cl}}(k) \le -\delta, \qquad \text{whenever } V(x_{\text{cl}}(k)) > \delta. \tag{26}$$

(v) The subset $\bar{B}(x^d, \epsilon_d) := \{x \in \mathbb{R}^n \mid \|x - x^d\|_\infty \le \epsilon_d\}$ is invariant and attractive under (25); furthermore, it is reachable in finite time.

Proof. It will be shown that V defined by

$$V(x) := \min_{i \in \{1,\ldots,n\}} \{t_i^{\text{opt}}(x) \,|\, X(t_i^{\text{opt}}(x); 0; x; U(\cdot, t^{\text{opt}}(x^0), u^{\text{opt}}(x^0))) = 0\} \tag{27}$$

satisfies the Properties (i) to (v) of Proposition 3.1. Note that, because of the steering property of $(t^{\text{opt}}(x), u^{\text{opt}}(x))$, we know that $V(x)$ is well defined and such that $V(x) \le t_n^{\text{opt}}(x)$. The definition (27) enables us to detect the early steering achievement. This is crucial in the following proof.

(i) This results directly from the evident fact that, under bounded control, the system state cannot be steered from $x^0 \ne x^d$ to x^d infinitely fast.

(ii) This results from the straightforward fact that, under bounded control, the time necessary to steer the state from x^0 to x^d tends to infinity when $\|x^0\|$ tends to infinity.

(iii) Let $\chi \subset \mathbb{R}^n$ be some bounded subset and let us prove that there is some $M > 0$ such that, for all $x \in \chi$,

$$V(x) = t_n^{\text{opt}}(x) \le M.$$

Note that, for all x,

$$V(x) \le q^0\delta + \hat{t}_n(x^f(x, q^0, \epsilon^0)) \le \delta + \hat{t}_n(x^f(x, q^0, \epsilon^0)), \tag{28}$$

where (q^0, ϵ^0) is any pair in $\{0, 1\} \times \{-1, 1\}$ such that $x_1^f(x, q^0, \epsilon^0) \ne x_1^d$. Now, since $x^f(x, q^0, \epsilon^0)$ belongs clearly to a bounded set (because $q^0 \le 1$ and, under constant u_1, z obeys a linear differential equation). Hence, according to (28), it is sufficient to prove that $\hat{t}_n(\cdot)$ is bounded over bounded sets. But one clearly has

$$\hat{t}_n(x) \le (n-1)\hat{\tau}(x),$$

where $\hat{\tau}(x)$ is the minimum τ necessary for $(\bar{t}(\tau), \bar{u}(\tau, x))$ to be admissible (see Lemma 2.1). Now, according to (9) and (15), $\hat{\tau}(x)$ meets the following inequality:

$$\hat{\tau}(x) \le \max\{|x_1 - x_1^d|/(n-1)u_1^{\max}, [\|\bar{u}_2(\tau_0, x)\|_\infty / u_2^{\max}]\tau_0\}, \tag{29}$$

where $\tau_0 > 0$ is arbitrary. Clearly, this ends the proof of (iii), since the r.h.s. of (29) is bounded over bounded subsets of $\mathbb{R}^n$.

(iv) Let us denote by $J(q, \epsilon, x^0)$ the cost function in (20), namely,

$$J(q, \epsilon, x^0) := q\delta + \hat{t}_n(x^f(x, q, \epsilon)), \tag{30}$$

and let us use $(\hat{q}_k, \hat{\epsilon}_k)$ to denote $(\hat{q}(x_{\text{cl}}(k)), \hat{\epsilon}(x_{\text{cl}}(k)))$. Two situations are to be handled:

Case 1. $\hat{q}_k > 0$. In this case, define

$$x^+(k) := X(\hat{q}_k\delta; 0; x_{\text{cl}}(k); u_{\hat{q}_k, \hat{\epsilon}_k}(\cdot)).$$

By definition, $V(x_{\mathrm{cl}}(k))$ is given by

$$V(x_{\mathrm{cl}}(k)) = \hat{q}_k\delta + \hat{t}_{i_0}(x^+(k)), \qquad \text{for some } i_0 \leq n. \tag{31}$$

Let us define the following suboptimal solution $(\tilde{q}_{k^+}, \tilde{\epsilon}_{k^+})$ of $P(x^+)$:

$$(\tilde{q}_{k^+}, \tilde{\epsilon}_{k^+}) := (\hat{q}_k - 1, \hat{\epsilon}_k).$$

It is then clear that

$$X(\tilde{q}_{k^+}\delta; 0; x_{\mathrm{cl}}(k^+); u_{\tilde{q}_{k+}, \tilde{\epsilon}_{k+}}(\cdot)) = x^+(k);$$

and therefore, by (31),

$$\begin{aligned} V(x_{\mathrm{cl}}(k^+)) &\leq J(\tilde{q}_{k^+}, \tilde{\epsilon}_{k^+}, x_{\mathrm{cl}}(k^+)) \\ &\leq (\hat{q}_k - 1)\delta + \hat{t}_{i_0}(x^+(k))) \\ &\leq V(x_{\mathrm{cl}}(k)) - \delta, \end{aligned}$$

which is nothing but (26).

Case 2. $\hat{q}_k = 0$. In this case, the next state on the closed-loop trajectory is clearly given by

$$x_{\mathrm{cl}}(k^+) = X(\delta; 0; x_{\mathrm{cl}}(k); U(\cdot, \hat{t}(x_{\mathrm{cl}}(k), \hat{u}^2(x_{\mathrm{cl}}(k))).$$

Now, choosing the suboptimal solution

$$(\tilde{q}_{k^+}, \tilde{\epsilon}_{k^+}) = (0, \hat{\epsilon}_k)$$

implies that

$$J(\tilde{q}_{k^+}, \tilde{\epsilon}_{k^+}, x_{\mathrm{cl}}(k^+)) = \hat{t}_n(x_{\mathrm{cl}}(k^+)), \tag{32}$$

but one clearly has

$$\hat{t}(x_{\mathrm{cl}}(k^+)) = \hat{t}(x_{\mathrm{cl}}(k)) - \delta, \tag{33}$$

$$\hat{u}(x_{\mathrm{cl}}(k^+)) = (\hat{u}^2(x_{\mathrm{cl}}(k^+)), \ldots, \hat{u}^n(x_{\mathrm{cl}}(k^+)), 0). \tag{34}$$

This with (32) shows clearly that, whenever $V(x_{\mathrm{cl}}(k)) > \delta$, the resulting steering time from $x_{\mathrm{cl}}(k)$ to x^d is lower by at least δ than the steering time from $x_{\mathrm{cl}}(k)$ to x^d. This clearly gives (26).

(v) The invariance of $\bar{B}(x^d, \epsilon_d)$ is straightforward, since according to the control law (25) $x \in \bar{B}(x^d, \epsilon_d)$ implies $\dot{x} = 0$. The facts that $\bar{B}(x^d, \epsilon_d)$ is attractive and is reachable in finite time are direct consequence of (26) and the already proved (i) of Proposition 3.1. □

4. Illustrative Examples

In this section, the real-time implementability of the proposed feedback law is first investigated. Then, some illustrative examples are proposed.

4.1. Real-Time Implementability. Recall that, in order to compute (25), one has to execute Algorithm A_{n-1} at most $2q_{\max}$ times. Recall also the crucial fact that Algorithm A_{n-1} is deterministic; therefore, an upper bound on the execution time can be obtained easily. Let t_{run} refers to this upper bound. Simulations on a DIGITAL-FORTRAN/PC-PENTIUM III 600MHZ shows that

$$t_{\text{run}} < 10^{-3} \text{ sec.} \tag{35}$$

By requiring that the execution time be five times lower than the sampling period δ, the following real-time implementability condition can be derived:

$$\delta \geq 10 \times q_{\max} \times t_{\text{run}} \sim 0.01 q_{\max}. \tag{36}$$

For instance,

$$q_{\max} = 10, \qquad \delta = 0.1$$

may be used.

4.2. Simulations. In all the following simulations, we use

$$q_{\max} = 10, \qquad \epsilon_d = 0.01, \qquad x^d = 0.$$

The simulations aim to illustrate the following facts:

(F1) When the admissible domain U_{ad} is enlarged, the resulting closed-loop stabilization time is reduced automatically. This can be seen on Figs. 1 and 2 where behaviors under two different saturation levels are shown.

(F2) The proposed feedback results in a unified behavior for both singular initial conditions $x_1^0 = x_1^d$ (Fig. 1) and nonsingular ones (Fig. 2). Indeed, in both cases, a first phase is initially engaged taking x_1 optimally far from x_1^d w.r.t. the minimum time steering objective; during this first phase, u_1 is constant and extremal while $u_2 = 0$. Then, the second stabilizing phase is engaged and $\bar{B}(x^d, \epsilon_d)$ is reached in finite time.

(F3) Because of truncation errors during numerical integration, the truly obtained steering time under closed-loop conditions may be greater than the one initially predicted by open-loop computations at the instant $t = 0$. This may be observed on Fig. 3. This confirms the closed-loop character of the control law insofar as it stabilizes the system in spite of the disturbances that may render the initial decisions out of order.

(F4) When the sampling period δ decreases, the resulting closed-loop stabilizing time decreases. This is closely related to Fact 4, since

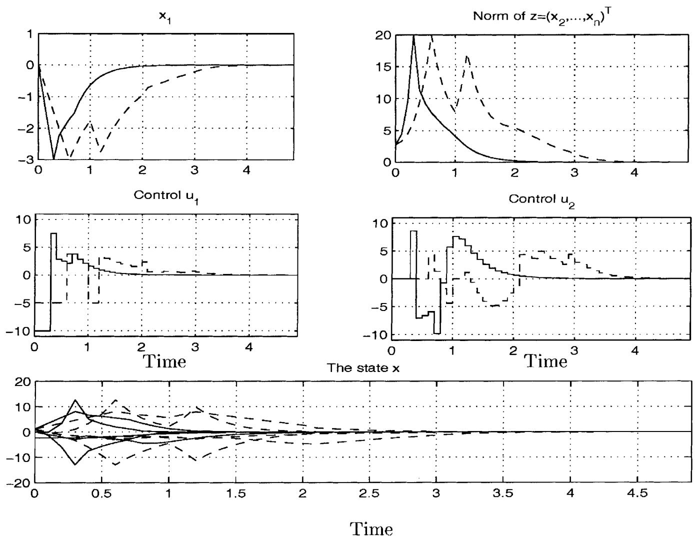

Fig. 1. Simulations in the singular case $x_0 = (0, -2.3, 1.2, 1.0, -0.1)^T$ with two different saturation levels $u_j^{\max} = 10$ (continuous line) and $u_j^{\max} = 5$ (dashed line).

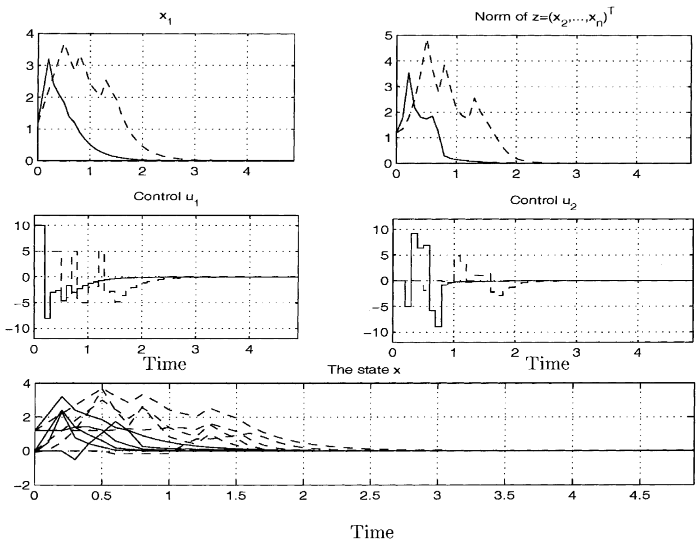

Fig. 2. Simulations in the nonsingular case $x_0 = (1.2, 0.0, 0.2, 0.0, -0.1)^T$ with two different saturation levels $u_j^{\max} = 10$ (continuous line) and $u_j^{\max} = 5$ (dashed line).

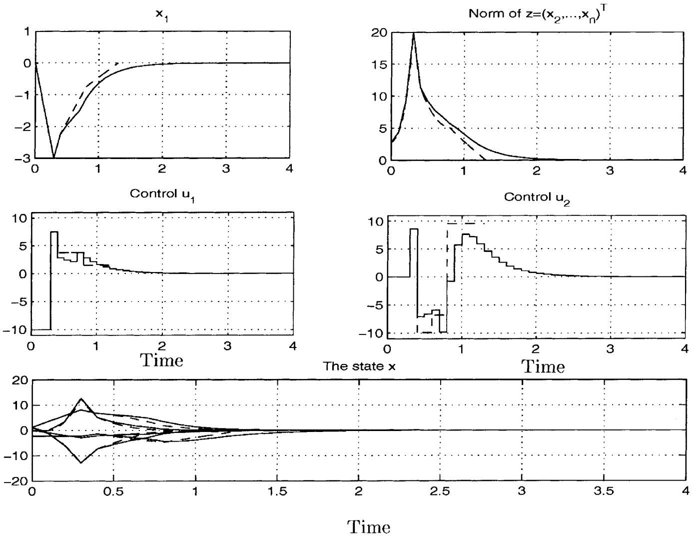

Fig. 3. Comparison between initial predicted open-loop trajectories (dashed line) and resulting closed-loop trajectories (continuous line) for $x_0 = (0, -2.3, 1.2, 1.0, -0.1)^T$ and saturation level $u_j^{\max} = 10$.

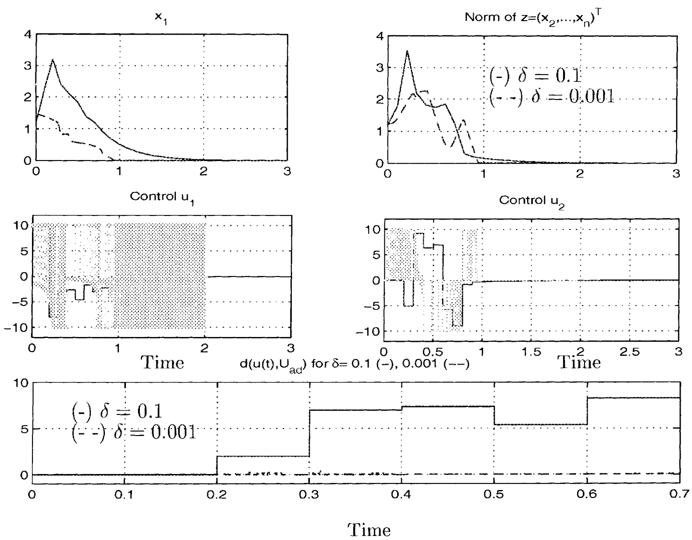

Fig. 4. Simulations with different sampling periods: $\delta = 0.1$ (continuous line) and $\delta = 0.001$ (dashed line). Note that the lower plot enables a numerical verification of $d(u(t), \partial U_{\mathrm{ad}}) = O(\delta)$.

it states that, when δ tends to zero, the applied control approaches the boundary ∂U_{ad} of the admissible domain, suggesting that the amplitude of the acceptable control is fully used to carry out the minimum time stabilizing objective. This can be observed on Fig. 4 and particularly on the lower plot, where the distance

$$d(u(t), \partial U_{ad}) = \min\{||u_1(t)| - u_1^{max}|, ||u_2(t)| - u_2^{max}|\}$$

is plotted vs. time. Note that simulations with $q_{max} = 10$ and $\delta = 0.001$ are given for illustrative purposes, despite the fact that these choices violate the real-time implementability constraint (36). Note that, apart from the above constraint, the behavior of the control for $\delta = 0.001$ is probably undesirable. However, the simulation enables Fact 4 to be verified numerically.

References

1. Murray, T. M., and Sastry, S., *Nonholonomic Notion Planning: Steering Using Sinusoids*, IEEE Transactions on Automatic Control, Vol. 38, pp. 700–716, 1993.
2. Brockett, R. W., *Asymptotic Stability and Feedback Stabilization*, Differential Goemetric Control Theory, Edited by H. J. Sussman, University of Michigan Press, Ann Arbor, Michigan, pp. 181–191, 1983.
3. Isidori, A., Nonlinear Control Systems, Springer Verlag, London, England, 1983.
4. Bloch, A., Reyhanoglu, M., and McClamroch, N. H., *Control and Stabilization of Nonholonomic Dynamical Systems*, IEEE Transactions on Automatic Control, Vol. 37, pp. 1746–1757, 1992.
5. Coron, J. M., *On the Stabilization in Finite Time of Locally Controllable Systems by Means of a Continuous Time-Varying Feedback Law*, SIAM Journal on Control and Optimization, Vol. 33, pp. 804–833, 1995.
6. Samson, C., *Control of Chained Systems: Application to Path Following and Time-Varying Point-Stabilization of Mobile Robots*, IEEE Transactions on Automatic Control, Vol. 40, pp. 64–77, 1995.
7. McCloskey, R. T., and Murray, R. M., *Exponential Stabilization of Driftless Nonlinear Control Systems Using Homogeneous Feedback*, IEEE Transactions on Automatic Control, Vol. 42, pp. 614–628, 1997.
8. Sørdalen, O. J., and Egeland, O., *Exponential Stabilization of Nonholonomic Chained Systems*, IEEE Transactions on Automatic Control, Vol. 40, pp. 35–49, 1995.
9. Godhavn, J. M., and Egeland, O., *A Lyapunov Approach to Exponential Stabilization of Nonholonomic Systems in Power Form*, IEEE Transactions on Automatic Control, Vol. 42, pp. 1028–1032, 1997.

10. Walsh, G., and Bushnell, L. G., *Stabilization of Multi-Input Chained Form Control Systems*, Systems and Control Letters, Vol. 25, pp. 227–234, 1995.
11. Teel, A. R., Murray, R. M., and Walsh, G. C., *Nonholonomic Control Systems: From Steering to Stabilization with Sinusoids*, International Journal of Control, Vol. 62, pp. 849–870, 1995.
12. Astolfi, A., *Discontinuous Control of Nonholonomic Systems.*, Systems and Control Letters, Vol. 27, pp. 37–45, 1996.
13. Alamir, M., and Bornard, G., *Globally Stabilizing Discontinuous Feedback Law for a Class of Nonlinear Systems*, European Journal of Control, Vol. 2, pp. 44–50, 1996.
14. Alamir, M., and Khennouf, H., *Discontinuous Receding-Horizon Control-Based Stabilizing Feedback for Nonholonomic Systems in Power Form*, Proceedings of the IEEE Conference on Decision and Control, New Orleans, Louisiana, 1996.
15. Luo, J., and Tsiotras, P., *Control Design of Chained Form Systems with Bounded Inputs*, Systems and Control Letters, Vol. 39, pp. 123–131, 2000.
16. Bloch, A. M., Drakunov, S. V., and Kinyon, M. K., *Stabilization of Nonholonomic Systems Using Isospectral Flows*, SIAM Journal on Control and Optimization, Vol. 38, pp. 855–874, 2000.
17. Kolmanovsky, I., Reyhanoglu, M., and McClamroch, N. H., *Switched Mode Feedback Control Laws for Nonholonomic Systems in Extended Power Form*, Systems and Control Letters, Vol. 27, pp. 29–36, 1996.
18. Jiang, Z. P., *Lyapunov Design of Global State and Output Feedback for Nonholonomic Control Systems*, International Journal of Control, Vol. 73, pp. 744–761, 2000.
19. Jiang, Z. P., and Nijmeijer, H., *Tracking Control of Mobile Robots: A Case Study in Back-Stepping*, Automatica, Vol. 33, pp. 1393–1399, 1997.
20. Fierro, R., and Lewis, F., *Control of a Nonholonomic Mobile Robot: Backstepping the Kinematics into Dynamics*, Proceedings of the 34th IEEE Conference on Decision and Control, New Orleans, Louisiana, pp. 3805–3810, 1995.
21. Jiang, Z. P., and Nijmeijer, H., *A Recursive Technique for Tracking Control of Nonholonomic Systems in Chained Form*, IEEE Transactions on Automatic Control, Vol. 44, pp. 265–279, 1999.
22. Fliess, M., Lévine, J., Martin, P., and Rouchon, P., *Design of Trajectory Stabilizing Feedback for Driftless Flat Systems*, Proceedings of the 3rd European Control Conference, Rome, Italy, pp. 1882–1887, 1995.