Further Results on Nonlinear Receding-Horizon Observers

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***Abstract*—In this note, further results are proposed that concern the design and the convergence of receding-horizon nonlinear observers. The key feature is the definition of observability radius in relation with a prespecified compact set of initial configurations. This enables a semiglobal convergence result to be derived that turns out to be a global convergence result when appropriate regularity assumptions are made. A simple example is proposed to illustrate the basic features of the note.**

***Index Terms*—Convergence results, nonlinear receding-horizon observers, observability radius, semiglobal.**

I. Introduction

The design of state observers for general nonlinear systems is still a difficult task. This is because of the high level concepts associated to the classical approaches and the related technical assumptions needed in the design procedures. An even short survey of existing approaches is clearly beyond the scope of this note. Surveys can be found in [1]–[3] and the reference therein. In this note, interest is focussed on receding-horizon observers [4]–[8]. Those are observers that use the integral output prediction error (IOPE) in the estimation process. The key advantage in using such observers is the absence of any kind of canonical forms to be exhibited in order for the corresponding observer to be designed. However, the associated computations are quite heavy and may prevent the use of these observers for systems with "*fast*" dynamics.

In [6], technical solutions have been proposed in order to partially overcome this drawbacks through the use of the poststabilization technique. It has been then proved that, in doing so, the ratio (precision/computation cost) can be greatly improved. The use of a regularized scheme to avoid singularities has been also proposed.

In [6], however, a global uniform regularity assumption has been used together with an a priori trajectories boundedness in order for convergence results to be derived. The aim of this note is to address the case where such assumptions are not assumed. Such global regularity assumption is typically used in many works that propose receding-horizon state estimation schemes like those proposed in [9]–[11] based on neural networks (see, for instance, [10, Ass. (ii)] and [11, Ass. A2]).

The result of the present note can be summarized as follows. Given a compact set S_0 of possible initial states, there is a high-gain receding-horizon observer that converges provided that the initial estimation error is lower than the observability radius associated to that compact set. It is also proved that under the assumption of global uniform regularity used in [6], the observability radius goes to infinity when the radius of S_0 goes to infinity. This enables the result of [6] to be recovered as a particular case.

The note is organized as follows. First, some definitions and notations are introduced in Section II. Then, the concept of observability radius is defined in Section III while Section IV presents the main result on semiglobal convergence. Finally, in Section V, an example is proposed to illustrate the different concepts developed throughout the note.

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II. Definitions & Notations

A. System's Related Definitions

Consider nonlinear controlled systems given by

$$\dot{x} = f(x,u) \quad y = h(x), \qquad u \in U \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and $u \in U \subset \mathbb{R}^m$ stand for the state, the measured output, and the control, respectively, and where f and h are supposed to be twice continuously differentiable. The control is supposed to belong to a compact set $U \subset \mathbb{R}^m$. It is also assumed that a piecewise constant control is used with some fixed sampling period $s > 0$ in order for the admissible control profiles $\tilde{u} \in U^{[t_0,t_0+Ns]}$ to belong to the finite-dimensional compact subset U^N of $\mathbb{R}^{Nm}$. Based on this assumption, subsets U^N and $\mathbb{R}^{[t-Ns,t]}$ are considered equivalent, and thus, used indifferently according to the context. $X(t;t_0;x_0;\tilde{u})$, $Y(t;t_0;x_0;\tilde{u}) := h(X(t;t_0;x_0;\tilde{u}))$ denote respectively the state and the output trajectories with initial conditions (t_0, x_0) under the control profile $\tilde{u} \in U^{[t_0,t]}$. Given a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, the notation $\varphi \in \mathcal{K}$ means that φ is continuous, strictly increasing and such that $\varphi(0) = 0$. For all vector functions $g(\cdot)$ defined over some time interval I, $\|g\|_{L_i}^I$ stands for $\int_I \|g(t)\|^i dt$ while $\underline{\sigma}(A)$ denotes the smallest singular value of the matrix A. $\rho(E)$ denotes the radius of the bounded set E (that is $\rho(E) := \sup_{x\in E} \|x\|$) with the convention $\rho(\emptyset) = \infty$. $d_2(E) := \{(x,x) | x \in E\}$ is the diagonal of $E \times E$. Finally, if E is some subset of an Euclidian space and $z \notin E$, $d(z,E)$ denotes the distance from z to E with the convention $d(z,\emptyset) = \infty$.

B. Observability Related Definitions

In [6], simple definitions for observability and detectability have been proposed and linked to existing ones [12]–[14]. For simplicity, only completely uniformly observable systems are considered in this note in order to concentrate on the main ideas.

Definition 1: [6]: System (1) is said to be completely uniformly observable if for all $T > 0$, there is $\varphi_1 \in \mathcal{K}$ such that for all $x^{(1)}$, $x^{(2)} \in \mathbb{R}^n$ and all admissible $\tilde{u} \in U^N$

$$\left\|x^{(1)} - x^{(2)}\right\| \le \varphi_1 \left(\left\| Y\left(.;t_0;x^{(1)};\tilde{u}\right) - Y\left(.;t_0;x^{(2)};\tilde{u}\right) \right\|_{L_2}^{[t_0,t_0+T]} \right). \tag{2}$$

C. Observer's Design Related Definitions

Consider some fixed horizon length $T = Ns > 0$. In what follows, $\tilde{u}_t$ denotes an admissible element in $U^{[t-T,t]}$ and $\tilde{u}_t(\tau)$ its value at $\tau \in [t-T,t]$. Suppose that during the system evolution, at each instant t, one disposes of the past output and control measurements over $[t-T,t[$. Using the notations $x^-(t) := x(t-T); u^-(t) := u(t-T)$, the following map may be defined:

$$\begin{aligned} J : \mathbb{R}^n \times \mathbb{R}^n \times U^N &\to \mathbb{R}^+ \left(z, x^-, \tilde{u}\right) \rightsquigarrow J\left(z, x^-, \tilde{u}\right) \\ &:= \left\| Y\left(.;0;z,\tilde{u}\right) - Y\left(.;0;x^-;\tilde{u}\right) \right\|_{L_2}^{[0,T]}. \end{aligned} \tag{3}$$

By the very definition 1 of observability, it is easy to recognize hat J is a key quantity in the observer design. This is because, according to (2), whatever the admissible $\tilde{u}$ is in U^N, $J(z,x^-,\tilde{u}) = 0$ would imply that $z = x^-$ and therefore that $x = X(T;0;z;\tilde{u})$. Hence, the observer design may be based on the following idea. Define $z(t)$ as the internal state of an observer; then provide $z(t)$ with a dynamics that forces $J(z(t), x^-(t), \tilde{u}_t)$ to decrease and take $\hat{x}(t) := X(T; 0; z(t), \tilde{u}_t)$ as the output of the observer.

The way the dynamics of z is defined in order to confer a decreasing behavior to J is based on a simple gradient descent strategy. That is why the gradient $G(z(t), x^-(t), \tilde{u}_t) := \partial J/\partial z(z(t), x^-(t), \tilde{u}_t)$ is a crucial quantity. In [6], G is supposed to be globally uniformally nonsingular. This, together with some additional technical boundedness assumptions, enabled a global convergence result to be obtained. Here, a different viewpoint is considered in order to better understand what happens in the absence of such global assumptions.

III. The Observability Radius

Rigourously speaking, the concept of observability radius developed in this section is partially linked to the use of gradient-based receding-horizon (GRH) observation scheme. It is then worth talking about the GRH observability radius. First of all, let us recall the following lemma.

Lemma 1: [Local Uniform Regularity of G] [6] : If for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ and all admissible $\tilde{u}_t \in U^{[t,\infty]}$, the system obtained from (1) by linearization around the trajectory that passes through (t, x), namely

$$\dot{\xi}(\tau) = \frac{\partial f}{\partial x}\left(X\left(\tau; t; x; \tilde{u}_t\right), \tilde{u}_t(\tau)\right)\xi(\tau);$$
$$v(\tau) = \frac{\partial h}{\partial x}\left(X\left(\tau; t; x; \tilde{u}_t\right)\right)\xi(\tau) \qquad (4)$$

is uniformly completely observable, then for all compact set $\mathcal{X} \subset \mathbb{R}^n$ there is a continuous strictly positive function $\alpha(\cdot)$ s.t $\underline{\sigma}(\partial^2 J/\partial z^2(x^-, x^-, \tilde{u}_t)) \ge \sigma(x^-) > 0$ for all $(t, x) \in \mathbb{R} \times \mathcal{X}$. ♣

The practical implication of Lemma 1 is the following. Given some fixed $(t, x^-) \in \mathbb{R} \times \mathcal{X}$ and some admissible $\tilde{u} \in U^N$, any singular point z of $J(., x^-, \tilde{u})$ is necessarily sufficiently far from x^-. More rigourously, we have the following result.

Corollary 1: Under the assumptions of Lemma 1, for all compact $\mathcal{X} \subset \mathbb{R}^n$ and all $\tilde{u} \in U^N$, (5), as shown at the bottom of the page, holds. This distance, denoted by $R_r(\mathcal{X})$ is referred to as the GRH regularity radius w.r.t the compact set $\mathcal{X}$. ♣

In other words, $R_r(\mathcal{X})$ is the least-possible distance $r > 0$ between x^- and *another* singular state z for $J(., x^-, \tilde{u})$.

Remark 1: It is worth noting that when the global uniform regularity of $G(., x^-, \tilde{u})$ is assumed as it is the case in [6], $R_r(\mathcal{X})$ defined by (5) is the distance from 0 to an empty set and, hence, $R_r(\mathcal{X}) = \infty$.

Remark 2: Remark 1 enables to strengthen the fact that GRH regularity radius does not represent radius around x^- beyond which linearization becomes invalid. In other words, z may still remain *within* the GRH regularity radius around x^- far away beyond the linearization's domain of validity. It really reflects how far the system is from the global uniform regularity assumption. (See the illustrative example in Section V for a quantitative illustration of this claim).

The GRH regularity radius gives the admissible norm of the estimation error before local singularities associated to a descent gradient approach become possible to encounter.

Related to $R_r(\mathcal{X})$ let us define the following two sets:

$$\mathcal{I}_r^{ad}(\mathcal{X}) := \left\{\left(z, x^-\right) \in \mathbb{R}^n \times \mathcal{X} | \left\|z - x^-\right\| \le R_r(\mathcal{X})\right\}$$
$$\partial\mathcal{I}_r^{ad}(\mathcal{X}) := \left\{\left(z, x^-\right) \in \mathbb{R}^n \times \mathcal{X} | \left\|z - x^-\right\| = R_r(\mathcal{X})\right\}.$$

According to the definition of the regularity radius $R_r(\mathcal{X})$, it is clear that there are no pairs (z, x^-) that are in $\mathcal{I}_r^{ad}(\mathcal{X})$ such that $z \ne x^-$ and $G(z, x^-, \tilde{u}) = 0$.

In relation with the subset $\partial\mathcal{I}_r^{ad}(\mathcal{X})$ previously defined, let us define the following quantity:

$$\bar{J}_r(\mathcal{X}) := \inf_{(z, x^-, \tilde{u}) \in \partial\mathcal{I}_r^{ad}(\mathcal{X}) \times U^N} J\left(z, x^-, \tilde{u}\right). \qquad (6)$$

It is easily seen that $\bar{J}_r(\mathcal{X}) > 0$, this readily comes from the compacity of the set $\partial\mathcal{I}_r^{ad}(\mathcal{X}) \times U^N$ over which minimization is performed and the uniform distinguishability of any pair (z, x^-) that belongs to $\partial\mathcal{I}_r^{ad}(\mathcal{X})$.

Remark 3: Again, if the global uniform regularity of G is assumed, pairs (z, x^-) in $\partial\mathcal{I}_r^{ad}(\mathcal{X})$ satisfy the following property:

$$\lim_{\rho(\mathcal{X}) \to \infty}\left[\inf_{(z, x^-) \in \partial\mathcal{I}_r^{ad}(\mathcal{X})} \|z - x^-\|\right] = \lim_{\rho(\mathcal{X}) \to \infty}[R_r(\mathcal{X})] = \infty$$

[see Remark 1] which, with (2), gives the following property:

$$\lim_{\rho(\mathcal{X}) \to \infty}\left[\bar{J}_r(\mathcal{X})\right] = \infty. \qquad (7)$$

The last definition needed to define the GRH observability radius is shown in (8) at the bottom of the page. This subset of $\mathcal{I}_r^{ad}(\mathcal{X})$ is clearly nonempty since it contains $d_2(\mathcal{X})$. Furthermore, since $J(., ., \tilde{u}) = 0$ over $d_2(\mathcal{X})$ for all $\tilde{u}$, $J_U^{-1}(\mathcal{X})$ contains an open set containing $d_2(\mathcal{X})$. With the aforementioned definitions in hand, the GRH observability radius can be defined.

Definition 2: Under the assumptions of Lemma 1, the GRH observability radius of (1) w.r.t the compact set $\mathcal{X}$ is given by (9), as shown at the bottom of the next page, where $J_U^{-1}(\mathcal{X})$ is defined by (8) ✠

The following straightforward proposition states the properties of the GRH observability radius that are of great interest in the main result of this note.

Proposition 1: The GRH observability radius given in Definition 2 is strictly positive. Furthermore, if the gradient G is globally uniformly regular then $\lim_{\rho(\mathcal{X}) \to \infty} R_O(\mathcal{X}) = \infty$ for all compact sets $\mathcal{X}$. ✠

IV. Convergence Results

The main result of this note is the following.

Proposition 2: Let $S_0 \subset \mathbb{R}^n$ be some compact set containing the origin and all initial conditions of interest. If the linearized system

$$d\left(0, \left\{r > 0 \quad \exists\left(x^-, z\right) \in \mathcal{X} \times \bar{B}\left(x^-, r\right) \setminus \left\{x^-\right\}, \quad \text{s.t.} \quad G\left(z, x^-, \tilde{u}\right) = 0\right\}\right) > 0. \qquad (5)$$

$$J_U^{-1}(\mathcal{X}) := \left\{\left(z, x^-\right) \in \mathcal{I}_r^{ad}(\mathcal{X}) \quad \text{s.t} \sup_{\tilde{u} \in U^N} J\left(z, x^-, \tilde{u}\right) \le \bar{J}_r(\mathcal{X})\right\}. \qquad (8)$$

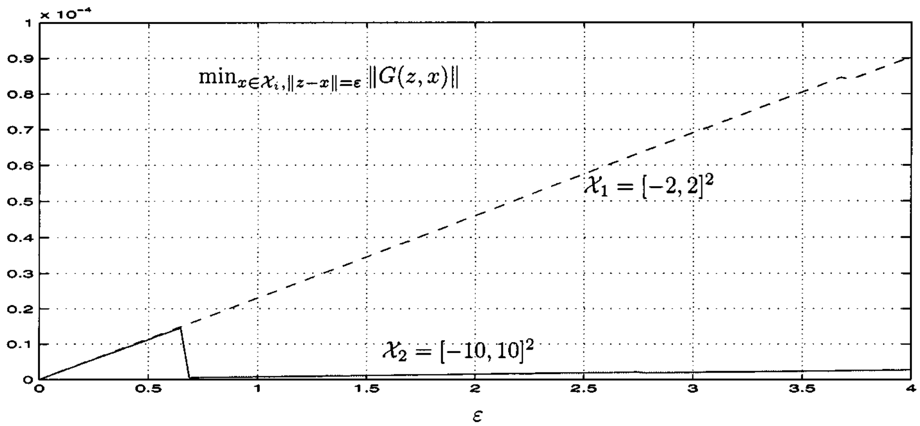

Fig. 1. Computation of the Regularity radius for system (13) for $\mathcal{X} = \mathcal{X}_1$ and $\mathcal{X} = \mathcal{X}_2$.

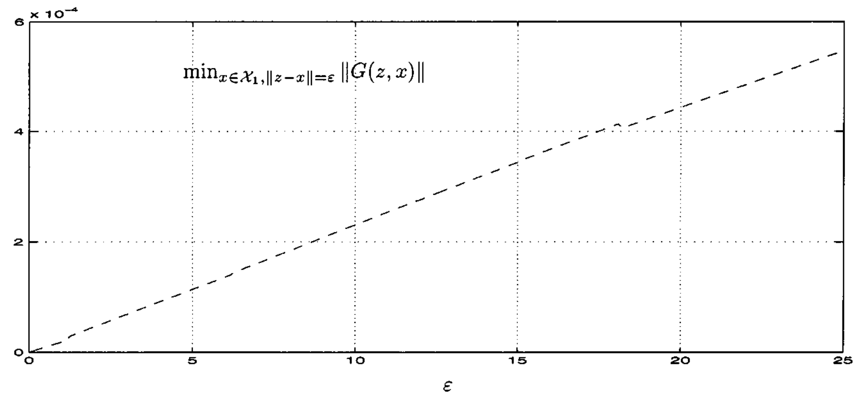

Fig. 2. Computation of the Regularity radius for system (13) for $\mathcal{X} = \mathcal{X}_1$.

(along admissible system's trajectories) is uniformly observable, then there is a sufficiently high $\gamma > 0$ for the dynamical system given by

$$\dot{z}(t) = f\left(z(t), u^-(t)\right) - \gamma G^T \left[GG^T\right]^{-1} \sqrt{J} \tag{10}$$

$$\hat{x}(t) = X\left(t; t-T; z(t); \tilde{u}_t\right) \tag{11}$$

(in which $G := G(z(t), x^-(t), \tilde{u}_t)$ and $J = J(z(t), x^-(t), \tilde{u}_t)$) to be a convergent observer for (1) for all initial configurations $(x^-, z) \in S_0 \times \mathbb{R}^n$ satisfying for some $\alpha > 1$

$$\left\|z - x^-\right\| < R_O\left(B\left(0, \alpha\rho\left(S_0\right)\right)\right). \tag{12}$$

In particular, if the gradient G is globally uniformly regular, then (10)–(11) is a globally convergent observer for (1). ✠

Proof: See the Appendix.

V. Illustrative Example

In this section, a simple example is given in order to illustrate the results and the definitions presented above. For, let us use the simple Van-der-pol oscillator already used in [6]

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = -9x_1 + 2\left(1 - x_1^2\right)x_2 \quad y = x_1. \tag{13}$$

Let us consider the two following choices for the set $\mathcal{X}$ of initial conditions:

$$\mathcal{X}_1 = [-2, 2] \times [-2, 2] \quad \mathcal{X}_2 = [-10, 10] \times [-10, 10].$$

The following three-variables function is then used to inspect the GRH observability radius of (13) w.r.t $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively:

$$F_\varepsilon(x, \theta) = \|G\left(z_\varepsilon(x, \theta), x\right)\| \quad z_\varepsilon(x, \theta) = x + \varepsilon\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}. \tag{14}$$

According to (5), computing the minimum of F_ε for $x \in \mathcal{X}$, $\theta \in [0, 2\pi]$ and for increasing $\varepsilon > 0$ enables one to compute either the GRH regularity radius $R_r(\mathcal{X})$ or a lower bound of it. (The IMSL library's subroutine DCBPOL is used to perform this optimization task). Figs. 1–2 show the evolution of the quantity

$$\min_{x\in\mathcal{X}, \theta\in[0,2\pi]} F_\varepsilon(x, \theta) = \min_{x\in\mathcal{X}, \|z-x\|=\varepsilon} \|G(z, x)\|$$

in the two cases, $\mathcal{X} = \mathcal{X}_1$ and $\mathcal{X} = \mathcal{X}_2$. From Figs. 1–2, it can be inferred that

$$R_r\left(\mathcal{X}_1\right) > 25; \quad R_r\left(\mathcal{X}_2\right) \approx 0.7$$

and, using these results, $\bar{J}_r(\mathcal{X}_i)$ can be computed according to (6) by minimizing $J(z_\varepsilon(x, \theta), x)$ (with $\varepsilon = R_r(\mathcal{X}_i)$) over $x \in \mathcal{X}$, $\theta \in$

$$R_O(\mathcal{X}) := \sup\left\{r < R_r(\mathcal{X}) \text{ such that } B\left(x^-, r\right) \subset J_U^{-1}(\mathcal{X}) \forall x^- \in \mathcal{X}\right\} \tag{9}$$

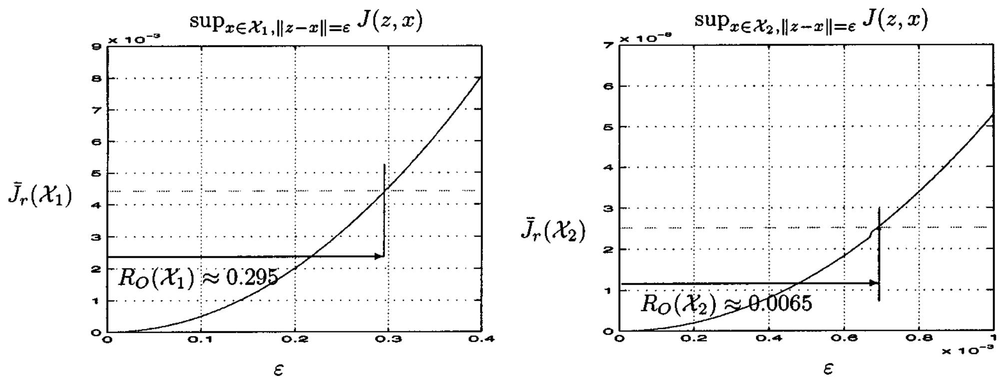

Fig. 3. Computation of the Observability radius for system (13) for $\mathcal{X} = \mathcal{X}_1$ and $\mathcal{X} = \mathcal{X}_2$.

$[0, 2\pi]$. This gives: $\bar{J}_r(\mathcal{X}_1) \approx 0.0044$; $\bar{J}_r(\mathcal{X}_2) \approx 0.25 \times 10^{-7}$. The GRH observability radius of (13) can then be evaluated according to Definition 2 for $\mathcal{X}_1$ and $\mathcal{X}_2$, respectively (see Fig. 3) to obtain

$$R_O(\mathcal{X}_1) \approx 0.295; \quad R_O(\mathcal{X}_2) = 0.0065.$$

APPENDIX
PROOF OF PROPOSITION 2

In the proof, the following notation is used:

$$\mathcal{X} := \bar{B}(0, \alpha\rho(S_0)). \tag{15}$$

In he proof, we shall prove the following three facts.

1) First, if the initial condition on $(z, x^-) \in \mathbb{R}^n \times S_0$ satisfies (12) then it is in $J_U^{-1}(\mathcal{X})$. Furthermore, for sufficiently high $\gamma > 0$, if (z, x^-) begins in $J_U^{-1}(\mathcal{X})$ then the trajectory remains in $\mathcal{I}_r^{ad}(\mathcal{X})$ during some time period $\Delta t(\mathcal{X}) > 0$.
2) As long as the trajectories remain in $\mathcal{I}_r^{ad}(\mathcal{X})$, the dynamic equation of the observers (10) is well defined and there is $\beta(\gamma) > 0$ such that
$$\frac{dJ}{dt} \le -\beta(\gamma)\sqrt{J} \tag{16}$$
where $\beta(\gamma)$ is a proper function of γ.
3) For sufficiently high γ, if the initial value of (z, x^-) is in $J_U^{-1}(\mathcal{X})$ then the solution of (16) tends to $J = 0$ in a finite time that is lower than $\Delta t(\mathcal{X})$.

The three facts cited above clearly enable to complete the proof. Let us begin by the simplest fact to prove, namely point 3). Indeed, it comes immediately by integrating (16) that $J^{1/2}(t) - J^{1/2}(t_0) \le -\beta(\gamma)/2(t - t_0)$ from which it becomes clear that $J(t)$ vanishes at some t, satisfying

$$t - t_0 \le \frac{2}{\beta(\gamma)} J^{1/2}\left(z(t_0), x^-(t_0), \tilde{u}\right) \le \frac{2}{\beta(\gamma)} \bar{J}_r(\mathcal{X}) \tag{17}$$

by definition of $J_U^{-1}(\mathcal{X})$ to which belongs (z, x^-) by assumption. It is then sufficient to choose γ sufficiently high so as to have $\beta(\gamma) > 2\bar{J}_r(\mathcal{X})/\cdot\sqcup(\mathcal{X})$. This ends the proof of point 3).

Let Us Now Prove 2)

According to the definition of $\mathcal{I}_r^{ad}(\mathcal{X})$, the gradient G is regular as long as $J \neq 0$. Therefore, the dynamic equation of J is well defined as long as the trajectory of (z, x^-) remains in $\mathcal{I}_r^{ad}(\mathcal{X})$ which is the assumption of 2). The proof of 2) is easily obtained by computing the derivative dJ/dt along the trajectory supposed to remain in the compact set $\mathcal{I}_r^{ad}(\mathcal{X})$

$$\begin{aligned}\frac{dJ}{dt}(t) = \int_0^T &\left[Y(\tau; 0; z(t), \tilde{u}) - Y\left(\tau; 0; x^-(t), \tilde{u}\right)\right]^T \\ &\times \left[\frac{\partial Y}{\partial z_I}(\tau; 0; z(t), \tilde{u})\, f\left(z(t), u^-(t)\right)\right. \\ &\left. - \frac{\partial Y}{\partial x_I}\left(\tau; 0; x^-(t), \tilde{u}\right) f\left(x^-(t), u^-(t)\right)\right] d\tau \\ &- \gamma\sqrt{J\left(z(t), x^-(t), \tilde{u}\right)}\end{aligned}$$

and, since $(z(t), x^-(t))$ remains in the compact set $\mathcal{I}_r^{ad}(\mathcal{X})$ as well as the admissible control profile in U^N, it is possible to find some $K > 0$ such that the last equation implies

$$\begin{aligned}\frac{dJ}{dt}(t) \le& K\int_0^T \left\|Y(\tau; 0; z(t), \tilde{u}) - Y\left(\tau; 0; x^-(t), \tilde{u}\right)\right\| d\tau \\ &- \gamma\sqrt{J\left(z(t), x^-(t), \tilde{u}\right)} \\ \le& \left(\frac{K}{T} - \gamma\right)\sqrt{J} =: -\beta(\gamma)\sqrt{J}\end{aligned}$$

with $\beta(\gamma) := \gamma - K/T$, which is clearly proper in γ. This ends the proof of 2).

Let Us Finally Prove 1)

$\surd$ First, it is worth noting that the initial state x^- belongs to S_0 that contains the origin. Therefore, the initial state x^- clearly belongs to $\mathcal{X}$ defined by (15). Furthermore, according to (12), we have that $z \in B(x^-, R_O(\mathcal{X}))$ and, hence, according to (9), that the initial state $(z(t_0), x(t_0)) \in int(J_U^{-1}(\mathcal{X}))$. Therefore, by definition

$$J\left(z(t_0), x^-(t_0), \tilde{u}\right) < \bar{J}_r. \tag{18}$$

$\surd$ Since the set of admissible controls is bounded and the initial state $x(t_0) \in S_0$, there is by definition of $\mathcal{X}$ [see (15)] a minimal time period $\Delta t(\mathcal{X})$ during which $x(t)$ remains in $\mathcal{X}$.

$\surd$ According to the definition of $\Delta t(\mathcal{X})$ and $\mathcal{I}_r^{ad}(\mathcal{X})$, it comes that during the time interval $[t_0, t_0+\Delta t(\mathcal{X})]$, the only way for the trajectory to leave $\mathcal{I}_r^{ad}(\mathcal{X})$ is that at some instant $t_1 \in]t_0, t_0 + \Delta t(\mathcal{X})[$ one has $(z(t_1), x^-(t_1)) \in \partial\mathcal{I}_r^{ad}(\mathcal{X})$. Let t_1^* be the first such instant, we shall prove that this leads to a contradiction.

$\surd$ Indeed, the minimality of t_1^* means that during the time interval $[t_0, t_1^*[$, one has $(z(t), x^-(t)) \in \mathcal{I}_r^{ad}(\mathcal{X})$. This would imply, according to point 2), that during this time interval, J satisfies the differential

(16), this, together with (18), implies that there is some $c > 0$ such that $J(z(t_1^*), x^-(t_1^*), \tilde{u}) \leq \bar{J}_r(\mathcal{X}) - c$ and this contradicts the fact that $(z(t_1^*), x^-(t_1^*)) \in \partial\mathcal{I}_r^{ad}(\mathcal{X})$ since the latter would imply, according to (6) that $J(z(t_1^*), x^-(t_1^*), \tilde{u}) \geq \bar{J}_r(\mathcal{X})$. This ends the proof of Proposition 2. Indeed, the global aspects are immediate consequences of Proposition 1. ◇

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Global Frequency Estimation Using Adaptive Identifiers

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***Abstract*—Online estimation of the frequencies of a signal which is the sum of n sinusoidals with unknown amplitudes, frequencies and phases is made through yet another well-known and simple system theoretical tool—adaptive identifiers. Convergence of the proposed estimator is proved. The new frequency estimator is of $3n$ order, as compared to the order $5n - 1$ resulting from Marino–Tomei observers. Results are demonstrated via simulation.**

***Index Terms*—Adaptive filter, adaptive identifier, frequency estimation, observer.**

I. INTRODUCTION

Consider the problem of online estimation of the frequencies $\omega_i > 0$, $i = 1, \ldots, n, \omega_i \neq \omega_j$, for $i \neq j$, of a signal of the following form:

$$y(t) = \sum_{i=1}^{n} A_i \sin(\omega_i t + \varphi_i) \tag{1}$$

where $y(t)$ is measurable, the amplitudes, $A_i \neq 0$, the phase angles, φ_i, are constant but also unknown. For simplicity, the signal in (1) is unbiasd. However, the technique to be developed can also be applied to a signal with an unknown constant bias.

Though this estimation problem is an important one in systems theory with applications in diverse fields [2], most of the existing solutions have been sought from the perspective of signal processing and/or telecommunication: line enhancers [14], finite impulse response filters [13], infinite impulse response filters or notch filters [7], [10], [11], and frequency locked loop [6]. They are also local. The first globally convergent estimator was proposed only recently in [3] for the case of a single frequency. This global estimator is based on the adaptive notch filter (ANF) and takes the following form:

$$\begin{aligned} \ddot{\xi} + 2\rho\hat{\omega}\dot{\xi} + \hat{\omega}^2\xi &= \hat{\omega}^2 y \\ \dot{\hat{\omega}} &= g\left(2\rho\dot{\xi} - \hat{\omega}y\right)\xi\hat{\omega} \\ g &= \frac{\epsilon}{\left\{1 + N\left[\xi^2 + \left(\frac{\dot{\xi}}{\hat{\omega}}\right)^2\right]\right\}(1 + \mu\,|\hat{\omega}|^\alpha)} \end{aligned} \tag{2}$$

with $\alpha > 1$ and ϵ, N and ρ positive reals.

The paper [3] has stimulated several responses from the control theoretical community. First, it was found in [15] that a simple fourth order estimator can be designed through the so-called Marino–Tomei observers for the case of a single frequency. Though the estimator is one order higher than the one given in [3], it has a simpler and more of a control system theoretical structure, as well as a more elegant global stability proof. Independently, [5] obtained the same result via designing an adaptive observer for the case of a single frequency and generalized the method to multiple frequencies with an unknown constant bias. It is noted that the order of this estimator is $5n - 1$ for the case of n frequencies. Another solution was provided by the application of a linear tracking differentiator [1].

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