Robust Stabilization of Nonlinear Systems by Discontinuous Dynamic State Feedback

M. Alamir, I. Balloul, N. Marchand

Laboratoire d'Automatique de Grenoble, France
email : Mazen.Alamir@lag.ensieg.inpg.fr

Abstract

In this paper, a discontinuous dynamic state feedback that robustly stabilizes affine in control uncertain nonlinear systems is proposed. The formulation is based on Hamilton-Jacobi-Isaacs partial differential equations with **two boundary conditions**. The resulting dynamic state feedback is then expressed in terms of the solution of the related PDE's. An interesting feature is that the internal state of the resulting dynamic state feedback may have discontinuous behavior as a function of time. The proposed scheme is illustrated through the example of the stabilization of the angular velocities of a rigid body under two actuators.

Keywords : Robust Stabilization, Nonlinear Systems, Unmeasured disturbances, Model uncertainty, Lyapunov function with jumps.

1 Introduction

Robust stabilization of general nonlinear systems is still an open problem and an exhaustive survey of the available results is beyond the scope of this paper.

Various approaches have been explored in the robust nonlinear control literature such as operator theory (Georgiou and Smith, 1997; Teel, 1996; Chen and Figueiredo, 1989), Lyapunov-based approaches (Qu, 1992; Najson and Kreindler, 1996) and variable structure method (Li *et al.*, 1995; Cong and Landers, 1995). The above references are naturally far from being exhaustive and a serious work is to be done in order to clearly summarize the main results in each of the preceding categories.

Much closer to our approach are the works that have been done in the context of nonlinear H_∞ control theory. This approach (Van der Schaft, 1991; Ball and Helton, 1992) attempts to generalize to nonlinear systems the concept of limited disturbance-to-output gain well known in the case of linear systems. The underlying mathematical tool for doing this is the two players zero-sum noncooperative games (Basar and Olsder, 1982).

Works in this approach generally consider nonlinear systems of the form $\dot{x} = f(x)+g(x)u+p(x)w$ where w stands for some unmeasured disturbance vector. The aim is then to design a feedback control that stabilizes the closed-loop system in the absence of disturbance (or for finite L_2-norm disturbances) under the zero-state detectability assumption **and** attenuates the effect of w on some regulated output z ; that is $\int_0^t z^T z dt \leq \gamma^2 \int_0^t w^T w dt$ and this for all t (where $\gamma > 0$ is some attenuation factor).

As a typical result in this context, one obtains a sufficient condition for the above problem to be solved. This sufficient conditions take the form of a partial differential equation (with final condition) on the state space called the Hamilton-Jacobi-Isaacs (**HJI**) equation. The solution $V(x)$ of this equation is then used to construct the static state feedback that solves the above problem.

This approach has been generalized to handle the measurement-based controller case where only the output measurements are available to construct the feedback (Ball *et al.*, 1993; Isidori and Astolfi, 1992; Lu and Doyle, 1994; James, 1995). This is generally done following roughly the same ideas that are used in the state feedback context but with some higher dimension system. Sufficient conditions of the same form are thus obtained and the solution of the corresponding PDE's is used to construct the static output feedback that solves the "*internal stability with disturbance attenuation*" problem.

In the above cited works, the main effort is concentrated on the disturbance attenuation goal. The stability of the resulting closed-loop system is then guaranteed for bounded L_2-norm disturbances $w(.)$ and only in the case where the system is zero-state detectable w.r.t the regulated output z. Alternatives with slight modifications have been proposed to explicitly handle the **robust stabilization problem**. In (Imura *et al.*, 1995), the H_∞ approach is combined with operator theory in order to design a robust stabilizing controller in the particular case of controller induced uncertainty. In (Savkin and Petersen, 1998), a nonlinear version of the bounded real lemma is used to convert the absolute stabilization problem into some equivalent H_∞ optimal control problem in the context of nonlinear discrete-time systems (see also (Savkin and Peterson, 1994)). This is done using **infinite horizon min-max problem's formulation** put into a dynamic programming form. The result (that generalizes a preceding work on time-varying linear uncertain systems (Savkin and Petersen, 1996)) is a necessary and sufficient condition for the above problem to be solved and the corresponding control is a dynamic state feedback. (The fact that makes the condition necessary in (Savkin and Petersen, 1998), comes from the assumption that the rejected disturbance is "*arbitrarily rich*").

The approach proposed in this paper is inspired by the following remarks

- It is *a-priori* justifiable to think that the problematic of disturbance rejection as expressed in the classical H_∞ formulations may not necessarily be the most suitable to properly handle the robust stabilization problem. Specific formulations might be proposed.
- In particular, the fact that the disturbance attenuation condition $\int_0^t z^T z dt \leq \gamma \int_0^t w^T w dt$ is imposed for all $t \geq 0$ contradicts a quite recurrent feature in the context of stabilization, namely, the fact that intermediate "*sacrifices*" are sometimes necessary to a final goal achievement. (This has already been pointed out in (Lin, 1996) where a point-wise attenuation condition $\int_0^T z^T z dt \leq \gamma \int_0^T w^T w dt$ is studied for some fixed T. However, this has not been carried through to derive a robust stabilization result).
- Regardless the relevance of the disturbance attenuation problem in the context of robust stabilization, it is quite interesting to note that the above problem has almost always been formulated without any boundedness condition on the disturbance level. This is naturally inspired by the generalization from the linear context, nevertheless, this might make the corresponding condition very restrictive in the nonlinear case. Moreover, this clearly corresponds to a quite unrealistic situation.

Following the above intuitions, a robust stabilization oriented formulation is proposed in this paper using the tools of noncooperative dynamic games theory. The basic idea is the following : Imagine that despite the bounded disturbances, there always exists an optimal min-max control strategy over some **finite horizon** that sensibly decreases some weighted norm of the state **at the end of the finite horizon** and this possibly through some intermediate "*sacrifices*" (note that this might not be possible with admissible unbounded disturbances). Assume also that this is true for all "*initial*" states that lies outside some neighborhood of the origin. Then, by repeated use of such end-point robust strategy, it is possible to steer the state of the system to the origin or at least to some neighborhood of the origin.

In this paper, the above intuitive idea is properly formulated to yield sufficient conditions and an associated **discontinuous dynamic state feedback**. The sufficient condition is expressed through a **time-varying partial differential equation with two boundary conditions**. The above PDE's has the classical form of a **HJI** equations. The reason why a time-varying **HJI** is needed can be easily understood from the above intuitive presentation where finite horizon time intervals are repeatedly used.

The approach proposed in this paper is a generalization of the results presented in (Marchand *et al.*, 2000) where the stabilisation of disturbance-free systems has been considered. In (Marchand *et al.*, 2000), it has been shown that the proposed approach enables a very good stabilisation capabilities to be reached especially for systems that do not meet the Brockett's conditions for C^1-stabilisability.

An underlying issue in all the above formulations is the solution of the corresponding partial differential equations. This is still an extremely hard task to which several works have been dedicated (see (Georges, 1996; James and Baras, 1996; Beard *et al.*, 1997) and more recently (Beard *et al.*, 1998; Beard and McLain, 1998; Alamir, 2001) and the references therein) and it constitutes in itself an active research area. In this sense, despite the fact that some academic examples can be given in which the related PDE's can be approximately solved (Isidori and Lin, 1998), the present work may legitimately inherit the same criticisms that can be levelled at almost all the above cited works on the H_∞-control formulations with respect to their possible applicability. On the other hand, however, it may benefit of all future breakthrough in the numerical solution of **HJI**-like partial differential equations.

This paper is organized as follows. First of all, the problem under consideration is properly stated in section 2. In section 3, the sufficient conditions are introdced together with an intuitive derivation of the feedback law. The basic result of the paper is given in section 4 while section 5 contains some illustrative examples.

Throughout the paper, the following notations are used. $\varphi \in \mathcal{K}$ refers to a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ that is continuous definite. If in addition, φ is proper ($\lim_{r\to\infty} \varphi(r) = \infty$), then the notation $\varphi \in \mathcal{K}_\infty$ is used. Given $r \geq 0$, $B_r \subset \mathbb{R}^n$ designates the open ball centred at the origin and of radius r while B_r^C stands for the complementary set of B_r, namely, $B_r^C := \{x \mid \|x\| \geq r\}$.

2 Problem Statement

Consider nonlinear systems given by :

$$\dot{x} = f(x) + g(x)u + p(x)w \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control and $w \in \mathbb{R}^p$ some unknown vector of disturbances and/or uncertainties. All the functions used in (1) are supposed to be continuous.

The unknown vector w is supposed to be bounded in the following sense

$$|w_i| \leq \bar{w}_i \quad ; \quad \text{for all } i = 1, \ldots, p \tag{2}$$

where the $\bar{w}_i$'s are a-priori given positive component-wise upper bounds. In what follows, $\bar{w}$ is used to denote the vector $[\bar{w}_1, \ldots, \bar{w}_p]^T \in \mathbb{R}^p$.

The commonly used notation $X(\tau; t_0, x_0, u, w)$ refers to the solution at instant τ of (1) that starts at (t_0, x_0) under the control $u(.)$ and the input $w(.)$.

The system (1) is supposed to meet the following natural assumption :

Assumption 1 [Natural condition on the nominal system]

There are two functions $\psi \in \mathcal{K}_\infty$ and $\beta \in \mathcal{K}$ such that for all initial state $x_0 \in \mathbb{R}^n$ and all $u(.)$, the following inequality holds :

$$\|x(t_f)\|^2 + \int_0^{t_f} \left[\|x(\tau)\|^2 + \|u(\tau)\|^2\right] d\tau \quad \geq \quad (1 + \beta(t_f))\psi\Big(\sup_{0 \leq \tau \leq t_f} \|x(\tau)\|\Big) \tag{3}$$

where $x(\tau) := X(\tau; 0; x_0; u; 0)$. ♡

Roughly speaking, this assumption simply states that the nominal system cannot be steered to the origin with no cost. In particular, large excursions correspond to high costs.

The aim of this paper is to propose a new feedback law enabling the origin of system (1) to be globally [resp. globally asymptotically] stabilized despite all uncertainties and/or disturbances satisfying (2).

3 Sufficient conditions and derivation of the feedback law

In this section, the sufficient conditions underlying the whole approach are first presented. Then the corresponding feedback law is intuitively introduced.

3.1 Statement of the sufficient conditions

There exist

- ⋄ *Three positive numbers* $T > 0$, $\rho_1 > 0$ *and* $\rho_3 > 0$,
- ⋄ *a continuous function* $\rho_2 : \mathbb{R}^n \to \mathbb{R}^+$,
- ⋄ *a* C^1 *function* $V : [0, T] \times \mathbb{R}^n \to \mathbb{R}$,
- ⋄ *an attenuation factor* $\gamma \in]0, 1[$ *and*
- ⋄ *a positive real* $r \geq 0$

such that *the two following conditions hold :*

1. $V(t, x)$ *satisfies the following PDE's with "terminal condition" for all* $(t, x) \in [0, T] \times \mathbb{R}^n$

$$V_t + V_x f(x) + \rho_3 x^T x + \frac{1}{4} V_x \big[\frac{1}{\rho_2(x)} p(x)p^T(x) - \frac{1}{\rho_1} g(x)g^T(x)\big] V_x^T = 0 \quad ; \quad V(T, x) = \|x\|^2 \tag{4}$$

2. $V(t, x)$ *satisfies the following "initial condition" for all* $x \in B_r^C$ *:*

$$V(0, x) + \bar{\rho}_2\Big(V(0, x)\Big) T \|\bar{w}\|^2 \leq \gamma \|x\|^2 \tag{5}$$

where $\bar{\rho}_2(y) := \sup\Big\{\rho_2(x) \quad | \quad \min_{\zeta_2 \in [0,T]} V(T - \zeta_2, x) \leq y\}$ ♡

3.2 Discussion: intuitive derivation of the feedback law

The following discussion aims to progressively introduce the proposed dynamic state feedback law. By the same, the justification of the above sufficient conditions is clarified. The following discussion also gives the basic arguments that enables the proof of the main result to be easily understandable almost without any additional effort.

- ✓ Suppose that system (1) satisfies the sufficient conditions defined in section 3.1.
- ✓ Using classical arguments from the (Hamilton-Jacobi-Isaacs)-related literature, it is well established that for all positive real $\zeta_2 \leq T$ and all t, $V\Big(T - \zeta_2, x(t)\Big)$ is the optimal value of the following dynamic game (Basar and Olsder, 1982) :

$$\min_{u(.)} \max_{w(.)} \Bigg\{ \|x(t + \zeta_2)\|^2 + \int_t^{t+\zeta_2} \Big[\rho_3 \|x(\tau)\|^2 + \rho_1 \|u(\tau)\|^2 - \rho_2(x(\tau)) \|w(\tau)\|^2\Big] d\tau \Bigg\} \tag{6}$$

where $x(\tau) := X(\tau; t; x(t); u; w)$.

✓ Furthermore, the associated optimal min-max control strategy is given by (Basar and Olsder, 1982) :

$$\hat{u}(\tau, x(\tau)) := -\frac{1}{2\rho_1} g^T(x(\tau)) V_x^T\Big(\tau - t + T - \zeta_2, x(\tau)\Big) \quad \forall \tau \in [t, t+\zeta_2] \tag{7}$$

✓ In particular, using $\zeta_2 = T$ in (6) shows that $V(0, x(t))$ is the optimal value of the dynamic game :

$$\min_{u(.)} \max_{w(.)} \left\{ \|x(t+T)\|^2 + \int_t^{t+T} \Big[\rho_3 \|x(\tau)\|^2 + \rho_1 \|u(\tau)\|^2 - \rho_2(x(\tau)) \|w(\tau)\|^2\Big] d\tau \right\} \tag{8}$$

and the optimal control strategy (7) becomes :

$$\hat{u}(\tau, x(\tau)) := -\frac{1}{2\rho_1} g^T(x(\tau)) V_x^T\Big(\tau - t, x(\tau)\Big) \quad ; \quad \tau \in [t, t+T] \tag{9}$$

✓ Now, from (8) and (5), it comes that the optimal trajectory under (9), say $\hat{x}(.)$ of the dynamic game (8) satisfies the following inequalities :

$$\sup_{t \le \tau \le t+T} \rho_2(\hat{x}(\tau)) \le \bar{\rho}_2\Big(V(0, x(t))\Big) \tag{10}$$

$$\|\hat{x}(t+T)\|^2 - \int_t^{t+T} \rho_2(\hat{x}(\tau)) \|w(\tau)\|^2 d\tau \quad \le \quad \gamma \|x(t)\|^2 - \bar{\rho}_2\Big(V(0, x(t))\Big) T \|\bar{w}\|^2 \tag{11}$$

✓ This with (11) prove that

$$\|\hat{x}(t+T)\|^2 \le \gamma \|x(t)\|^2 \tag{12}$$

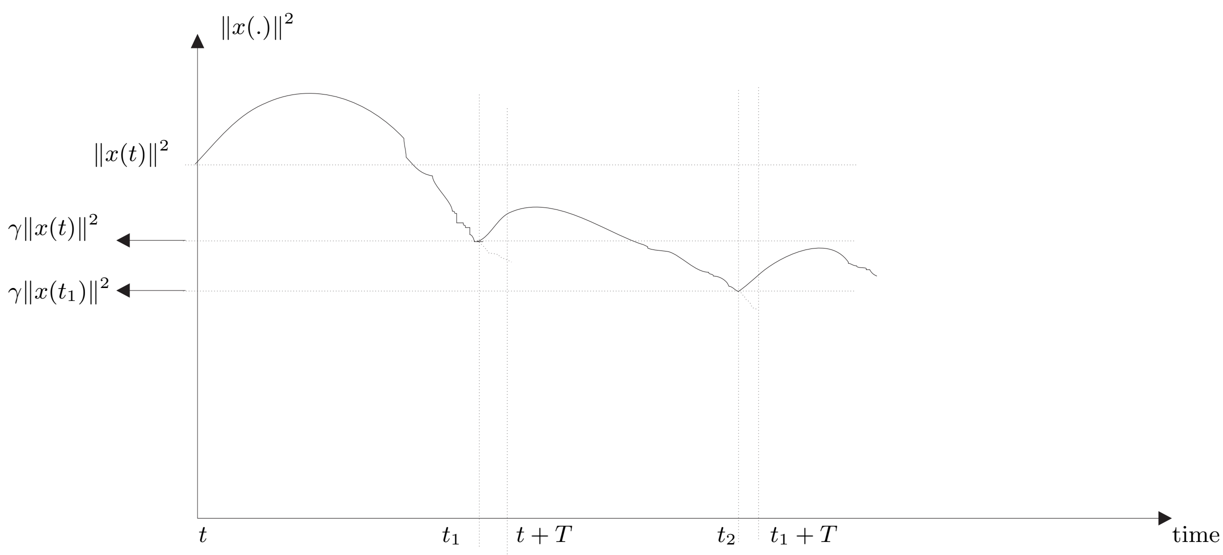


Figure 1: Generic behaviour of the closed-loop system

♯ **Summary** (see Figure 1) :

Starting from $(t, x(t))$ and applying the control strategy (9) for $\tau \ge t$, there is some $t_1 \le t+T$ such that $\|x(t_1)\|^2 \le \gamma \|x(t)\|^2$.

✓ Repeating the above argumentation from the new starting point $(t_1, x(t_1))$ yields a control strategy that steers the system's state to $(t_2, x(t_2))$ with :

$$\|x(t_2)\|^2 \le \gamma \|x(t_1)\|^2 \le \gamma^2 \|x(t)\|^2$$

and so on.

- ✓ The above ends the introduction of the main idea. However, some technical, but crucial details are still to be given to lead to a true dynamic state feedback.
- ✓ Let us call the instants t_i introduced above the "*commutation instants*".
- ✓ According to the discussion above, the optimal feedback control is defined over the time-intervals (t_i, t_{i+1}) by [see (9)] :

$$\hat{u}(\tau, x(\tau)) := -\frac{1}{2\rho_1} g^T(x(\tau)) V_x^T\Big(\tau - t_i, x(\tau)\Big) \quad ; \quad \tau \in]t_i, t_{i+1}[\tag{13}$$

- ✓ Using the fact that only the difference $\tau - t_i$ appears in (13), the following is clearly an equivalent implementation of (13) :

$$\dot{\zeta}_2(\tau) = -1 \quad ; \quad \zeta_2(t_i) = T \tag{14a}$$

$$\hat{u}(\tau, x(\tau)) := \hat{u}(\zeta_2(\tau), x(\tau)) = -\frac{1}{2\rho_1} g^T(x(\tau)) V_x^T\Big(T - \zeta_2(\tau), x(\tau)\Big) \tag{14b}$$

- ✓ Therefore, over (t_i, t_{i+1}), the optimal control is given by the dynamic state feedback (14) in which ζ_2 is an internal state of the controller that is to be re-initialized at the commutation instants t_i.
- ✓ The question is : How does the controller recognize whether an instant t^* is a commutation instant t_{i+1} or not ?
- ✓ This can be done by introducing a new internal state ζ_1 defined by :

$$\dot{\zeta}_1 = 0 \quad ; \quad \zeta_1(t_i) = \|x(t_i)\|^2 \tag{15}$$

  Indeed, with this, a commutation instant t^* is characterized by the following condition :

$$\|x(t^*)\|^2 \leq \gamma\zeta_1 \tag{16}$$

  roughly speaking, ζ_1 serves as a buffer store for the definition of t_{i+1}.
- ✓ Now, writing $\zeta := (\zeta_1, \zeta_2)^T \in \mathbb{R}^2$ and using the following definition :

$$C(x, \zeta) := \|x\|^2 - \gamma\zeta_1 \tag{17}$$

  the dynamic state feedback is given by :

$$\dot{\zeta} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{whenever } C(x,\zeta) > 0 \tag{18a}$$

$$\zeta = \begin{pmatrix} \|x\|^2 \\ T \end{pmatrix} \quad \text{whenever } C(x,\zeta) \leq 0 \tag{18b}$$

$$u(x, \zeta) = -\frac{1}{2\rho_1} g^T(x) V_x^T\Big(T - \zeta_2, x\Big) \tag{18c}$$

- ✓ There is still a final detail to be presented. For, let us go back to what happens on the time-interval (t_i, t_{i+1}) under the control law (14). Consider some instant $\tau \in (t_i, t_{i+1})$. The question is the following : what does guarantee the existence of a commutation instant $t_{i+1} \leq t_i + T$ satisfying $\|x(t_{i+1})\|^2 \leq \gamma\|x(t_i)\|^2$?

  In the absence of disturbances **other than** w, the answer is clearly contained in the sufficient condition (5) and the use of the corresponding optimal strategy.

✓ In the presence of other unpredictable disturbances, however, this may not still be the case. It is then important to detect such disturbances and to re-initialize the controller's internal states ζ_1 and ζ_2. For, the following condition is used which is a dynamic version of (5) :

$$V(T-\zeta_2, x(\tau)) + \bar{\rho}_2\Big(V(T-\zeta_2, x(\tau))\Big)\zeta_2\|\bar{w}\|^2 < \bar{\gamma}\zeta_1 \quad \text{where } \bar{\gamma} := \tfrac{\gamma+1}{2} \in]\gamma, 1[\tag{19}$$

(note that the use of $\bar{\gamma}$ instead of γ as it is done in (5) enables a strict inequality to be used and avoid false alarms to be launched due to numerical truncation errors). Indeed, if (19) is not satisfied, the controller should decide that unpredictable and harmful disturbances took place on $]t_i, \tau)$ rendering the final objective unfeasible and justifying the need to a re-initialisation of the controller's internal state.

Roughly speaking, condition (19) is somehow a receding-horizon feasibility test that coincides with (5) at the beginning of the time interval between two commutation instants (where $\zeta_2 = T$).

✓ Finally, the feedback law is given by (18) in which, $C(x, \zeta)$ is changed to

$$C(x,\zeta) := \min\left\{\|x\|^2 - \gamma\zeta_1 \ , \ \bar{\gamma}\zeta_1 - V(T-\zeta_2, x) - \bar{\rho}_2\Big(V(T-\zeta_2, x)\Big)\zeta_2\|\bar{w}\|^2\right\} \tag{20}$$

Indeed, commutation takes place when $C(x,\zeta) \le 0$ that is, when one of the following occurs :

- The goal is reached ($\|x\|^2 \le \gamma\zeta_1$),
- The goal is unfeasible without a new re-initialisation.

3.3 Some remarks on the feedback law

♮ The structure of feedback defined by (18) and (20) can be viewed as a generalization of the sliding-mode controller's structure in which $C(x, z) = 0$ plays the role of a sliding surface.

However, two differences have to be underlined :

1. In the proposed law, the structures of the equations are deeply different on the two sides of the surface $C(x,\zeta) = 0$. Indeed, for $C(x,\zeta) > 0$, one obtains a dynamic state feedback while for $C(x,\zeta) \le 0$, a static state feedback is used.
2. The above difference has a nice consequence on the closed-loop behaviour. Indeed, after a commutation that occurs at instant t_i, one has

$$C(x(t_i^+), \zeta(t_i^+)) = (1-\gamma)\|x(t_i)\|^2 > 0 \tag{21}$$

and the time necessary for C to become negative is a positive definite function of $\|x\|$. The consequence of this is that **the solution of the closed-loop system equations is well defined in a conventional frame**. There is no need to invoke solutions in the filippov sense.

♮ The use of a state dependent weighting function $\rho_2(x)$ in the expression of the sufficient condition (4)-(5) enables an asymptotic stabilisation to be potentially obtained. Indeed, if a constant ρ_2 has been used, the "*initial condition*" (5) would be the following :

$$V(0, x) + \rho_2 T\|\bar{w}\|^2 \le \gamma\|x\|^2 \tag{22}$$

and this is impossible to hold for $x \to 0$. Therefore, a constant ρ_2 makes impossible an asymptotic stability result even when this can be legitimately expected (for instance when $p(0) = 0$).

4 Main results

At present, the main result of this paper can be stated :

Theorem 1 [Robust stabilization]

Provided that *system (1) satisfies the natural assumption 1 and under the sufficient conditions defined in section 3, the subset :*

$$\mathcal{A}_r := \left\{ x \in \mathbb{R}^n \quad | \quad \| x \| \leq \psi^{-1}\Big(\frac{\gamma r^2}{(1+\beta(T))\min(\rho_1,\rho_3)}\Big) \right\} \tag{23}$$

is globally asymptotically stable w.r.t the closed-loop system's dynamic associated to the feedback law defined by :

$$\dot{\zeta} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{whenever } C(x,\zeta) > 0 \tag{24a}$$

$$\zeta = \begin{pmatrix} \Omega(x,r) \\ T \end{pmatrix} \quad \text{whenever } C(x,\zeta) \leq 0 \tag{24b}$$

$$u(x,\zeta) = -\frac{1}{2\rho_1} g^T(x) V_x^T\Big(T - \zeta_2, x\Big) \tag{24c}$$

where :

$$\Omega(x,r) = \begin{cases} \|x\|^2 & \text{if } \|x\| \geq r \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

In particular, if *the sufficient conditions are satisfied with $r = 0$,* **then** *the feedback law globally asymptotically stabilizes the origin $x = 0$.*

Both cases *correspond to bounded evolutions of the controller's internal states.* ♠

Note that in the case where $r \neq 0$, equation (24b) slightly differs from the expression (18b) of section 3. This comes from the fact that all the arguments used in the discussion that are based on (5) are valid only outside B_r. Therefore, the possibility of future decrease in the state's norm cannot be guaranteed for $\|x\| < r$. That is why there is no reason to set ζ_1 to $\|x\|^2$ in this case. Furthermore, setting ζ_1 to 0 transforms the used feedback into a finite-horizon constraint-free classical feedback which is a quite nice feature. This is not however the only possible choice for the definition of the feedback inside B_r. Other choices can be taken. For instance, a locally stabilizing feedback can be used.

PROOF: see appendix.

5 Example

As an illustrative example, let us consider the under-actuated satellite given by the following equations :

$$\begin{aligned} \dot{\omega}_1 &= a_1\omega_2\omega_3 + u_1 + b_1 w \\ \dot{\omega}_2 &= a_2\omega_3\omega_1 + u_2 + b_2 w \\ \dot{\omega}_3 &= a_3\omega_1\omega_2 + b_3 w \end{aligned}$$

where the ω_i's are the angular velocities and the a_i's are the inertial characteristics w.r.t the principal axis of inertia.

The stabilization of the above system using only two actuators simulates the case where the third actuator is out of order. It is somehow a classical problem in the nonlinear control community (Brockett *et al.*, 1983; Aeyels and Szafranski, 1988; Sontag and Sussman, 1988; Outbib and Sallet, 1992; Outbib, 1994). All the works cited above tackle only free disturbance models. More recently, a robust stabilizing feedback has been proposed (Astolfi and Rapaport, 1998) based on inverse optimality design. More precisely, in (Astolfi and Rapaport, 1998), a robust stabilizing controller is proposed based on a modified version of the classical HJI's inequalities to take into account non differentiable solutions of the above inequalities. To do so, the generalized gradient of Clarke is invoked (Clarke, 1990). However, these inequalities are still classical since they reflect unbounded disturbances attenuation with internal stability. That is, no explicit treatment is possible when an upper bound is given on the disturbance input's level. An interesting fact shown in (Astolfi and Rapaport, 1998) (see proposition 3) is that when $b_3 \neq 0$, one has to exclude a neighborhood of the origin in order to find differentiable solutions of the HJI's inequality. This may recall the role played by the set B_r in the present paper.

Numerical values used in the following simulations are the following

$$a_1 = -0.25 \quad ; \quad a_2 = 0.5 \quad ; \quad a_3 = -1$$

The numerical solution of the PDE's with boundary conditions (4)-(5) is done using the subroutine "*NumSol*" (Balloul, 2000). This subroutine is based on a classical approach that uses functional approximation basis whose coefficients are obtained by a collocation method.

5.1 Free-disturbance simulations

First of all, simulations are proposed for the nominal free-disturbance system ($w_i \equiv 0$). The PDE's (4)-(5) defining the sufficient condition have been solved using the following set of parameters :

$$\gamma = 0.99 \quad ; \quad \rho_1 = 0.1 \quad ; \quad \rho_2(x) = Cste = 1 \quad ; \quad \rho_3 = 10^{-6}$$

note that $\rho_2(x)$ is chosen to be constant since $p(x)$ is also state-independent. Solution has been obtained in which $T = 2.12$. Figure 2 shows the closed-loop behaviour in this case. The evolutions of the system's state as well as the dynamic controller's states are shown on Figure 2(a) while controls are shown on Figure 2(b). Note the discontinuous nature of the control profiles and the characteristic behaviour of $\|\omega\|^2$ that is in the heart of the proposed stabilization scheme.

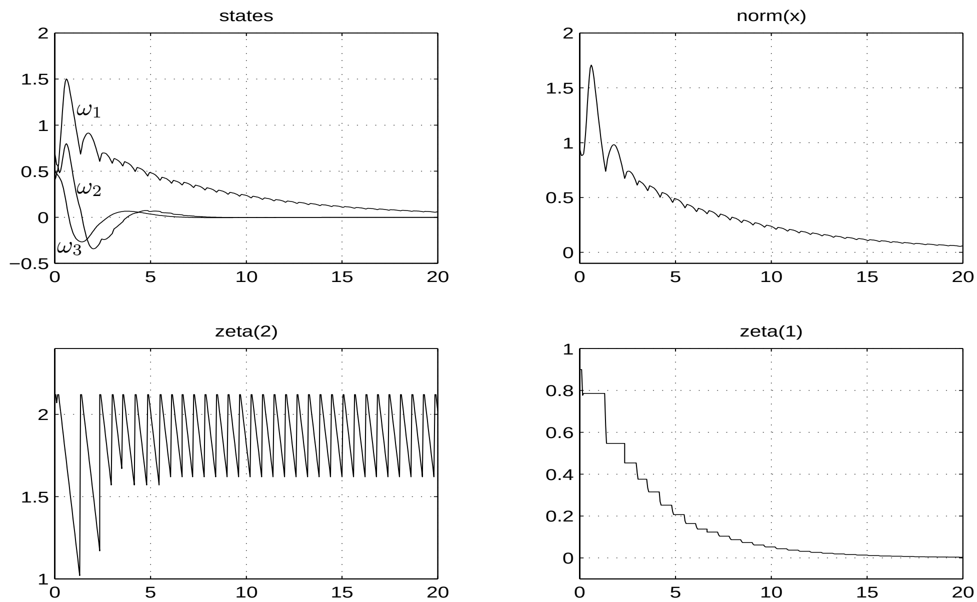
On Figure 3, a comparison with some existing law is proposed. The parameters of the different laws have been tuned in order to obtain roughly the same control levels. The proposed law seems to behave slightly better in steering the state to the origin. The control however is quite "*nervous*" as it can be expected from the theory.

Since the feedback proposed in (Astolfi and Rapaport, 1998) is defined by 9 parameters, namely σ_1, σ_2, γ, α, β, δ, c_1, c_2 and c_3, it seems difficult to show completely fair comparison. The following choices have been made (see (Astolfi and Rapaport, 1998) for parameters signification)

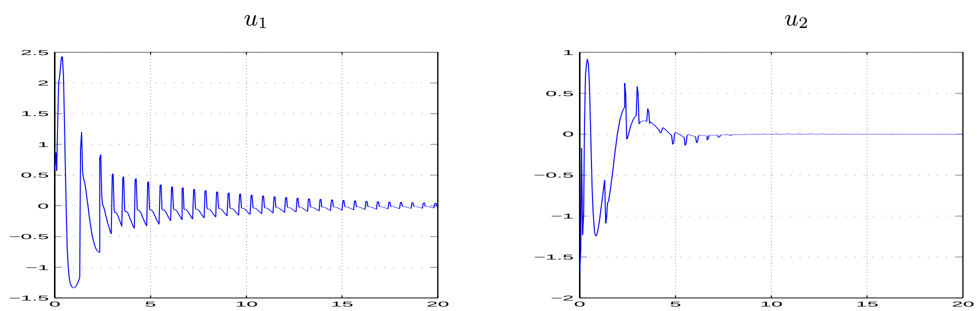
- ✓ $c_1 = c_2 = c_3 = 1$, namely, the regulated variable is $z = x$.
- ✓ $\beta = 1$, $\alpha = 1$ and $\delta = 1 + 1/|A| > \delta_{min} := 1/|A|$. Since no special recommendations are given to orient this choice.
- ✓ For the remaining parameters σ_1, σ_2, γ, the following is done
  - ♮ First, a choice $(\sigma_1^{(0)}, \sigma_2^{(0)}, \gamma^{(0)})$ is found that leads to the same control maximal excursion than the one obtained by the feedback proposed in this paper.
  - ♮ Starting from this first choice, two other parameterizations are tested, namely

$$\begin{aligned}(\sigma_1^{(1)}, \sigma_2^{(1)}, \gamma^{(1)}) &:= (\sigma_1^{(0)} \times 10^4, \sigma_2^{(0)} \times 10^4, \gamma^{(0)}) \\ (\sigma_1^{(2)}, \sigma_2^{(2)}, \gamma^{(2)}) &:= (\sigma_1^{(0)}, \sigma_2^{(0)}, \gamma^{(2)}) \quad \gamma^{(2)} \ll \gamma^{(0)}\end{aligned}$$

In order to understand the relevance of the above parameterization, recall that according to (Astolfi and Rapaport, 1998), increasing σ_1 and σ_2 reduces the radius of the attractive neighborhood of the origin while γ is the resulting attenuation factors appearing in the famous robustness inequality $\| z \|_2^2 \leq \gamma^2 \| w \|_2^2$.



(a) States



(b) Controls

Figure 2: Stabilization of the angular velocities of the rigid body. The free-disturbance case

In the case of free-disturbance context, the following parameterizations are tested for the feedback law proposed in (Astolfi and Rapaport, 1998)

Parametrization 1: $(\sigma_1^{(0)}, \sigma_2^{(0)}, \gamma^{(0)}) := (4, 4, 0.9)$

Parametrization 2: $(\sigma_1^{(1)}, \sigma_2^{(1)}, \gamma^{(1)}) := (40000, 40000, 0.9)$

Parametrization 3: $(\sigma_1^{(2)}, \sigma_2^{(2)}, \gamma^{(2)}) := (4, 4, 0.1)$

The results are given on Figure 4 from which the same conclusions can be drawn, namely the best capacity of the proposed feedback to bring back the state to the origin at the price of a more nervous control behaviour. Note that the control inputs corresponding to parameterization 2 have not be drawn since values of about 10^6 are obtained.

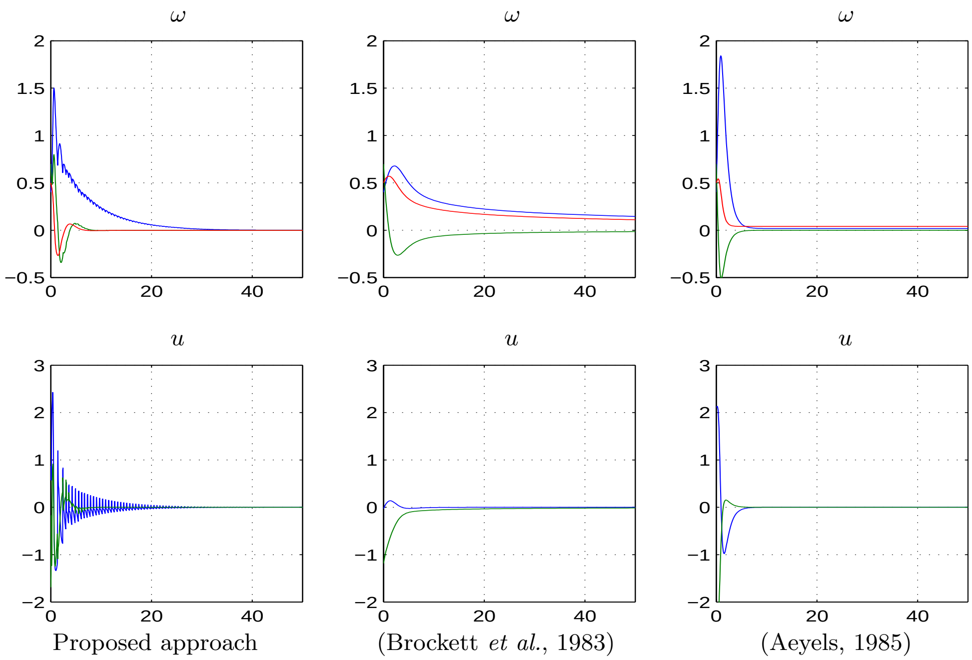


Figure 3: Comparison with existing law. Free-disturbance case

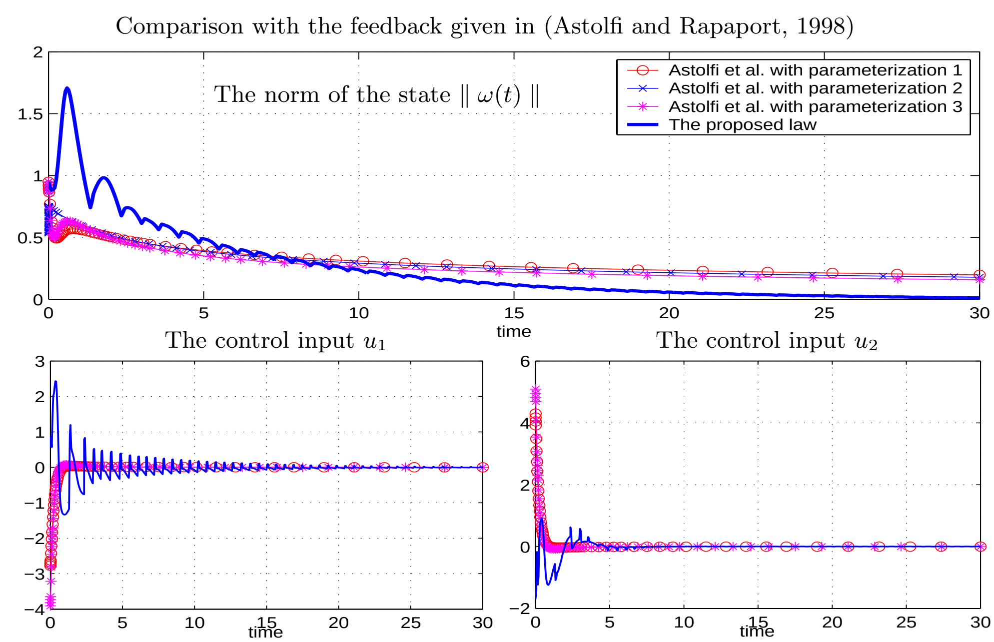


Figure 4: Comparison with Some existing laws. Free-disturbance case

5.2 Simulation in presence of disturbances

For these simulation, sinusoïdal signal of amplitude 1 and period 2.5 seconds has been considered for w while the following parameters have been used for the control :

$$\gamma = 0.9 \quad ; \quad \rho_1 = 0.01 \quad ; \quad \rho_2(x) = Cste = 0.1 \quad ; \quad \rho_3 = 0.01 \quad ; \quad \|\bar{w}\| = 1$$

Solution has been found with $T = 0.804$.

Figures 5-6 show the behaviour of the closed-loop system for different values of the gains b_i's.

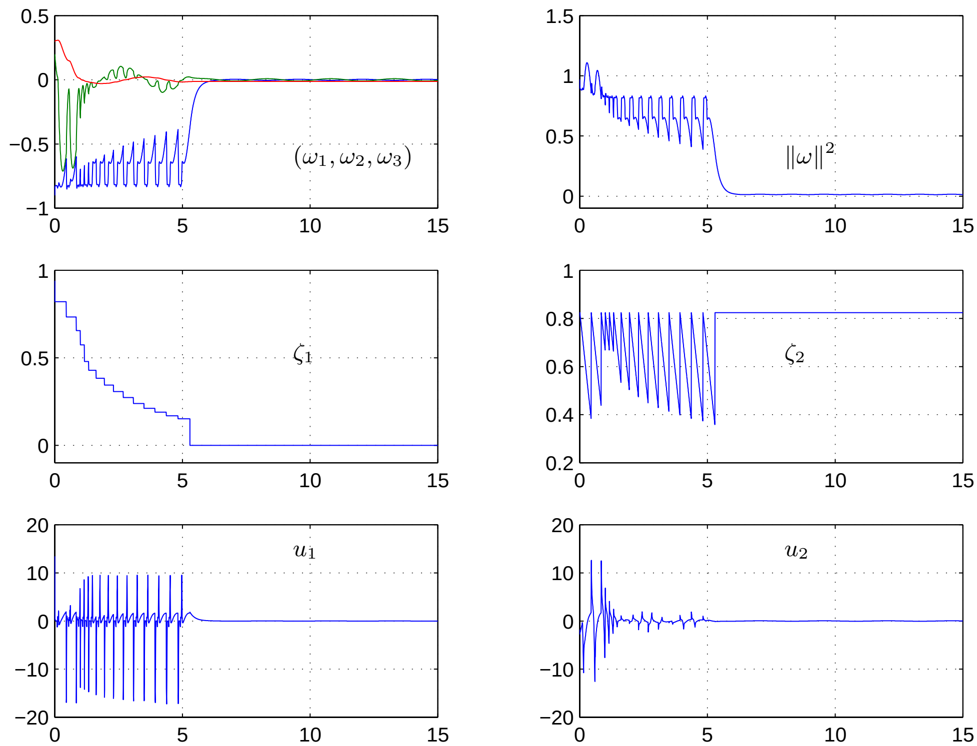


Figure 5: Disturbed case: $b_1 = 0.1a_1$, $b_2 = 0.1a_2$ and $b_3 = 0$

As for the comparison with the feedback proposed in (Astolfi and Rapaport, 1998), the same procedure explained above has been used to obtain the following parameterizations

Parametrization 3: $(\sigma_1^{(0)}, \sigma_2^{(0)}, \gamma^{(0)}) := (20, 20, 0.9)$

Parametrization 4: $(\sigma_1^{(1)}, \sigma_2^{(1)}, \gamma^{(1)}) := (200000, 200000, 0.9)$

Parametrization 5: $(\sigma_1^{(2)}, \sigma_2^{(2)}, \gamma^{(2)}) := (20, 20, 0.1)$

The results are shown on Figure 7 (note again that the control inputs corresponding to parameterization 5 have not be drawn since values of about 10^6 are obtained). Again, the proposed feedback law shows better performances in stabilization and disturbance rejection at the price of nervous control inputs.

6 Conclusion and future work

In this paper, a novel approach for the design of robust stabilising feedback has been proposed. The feedback so-obtained is a discontinuous dynamic state feedback that is derived from the solution of a Hamilton-Jacobi-Isaacs like PDE with two boundary conditions. Although the stabilizing capabilities of the proposed law seem

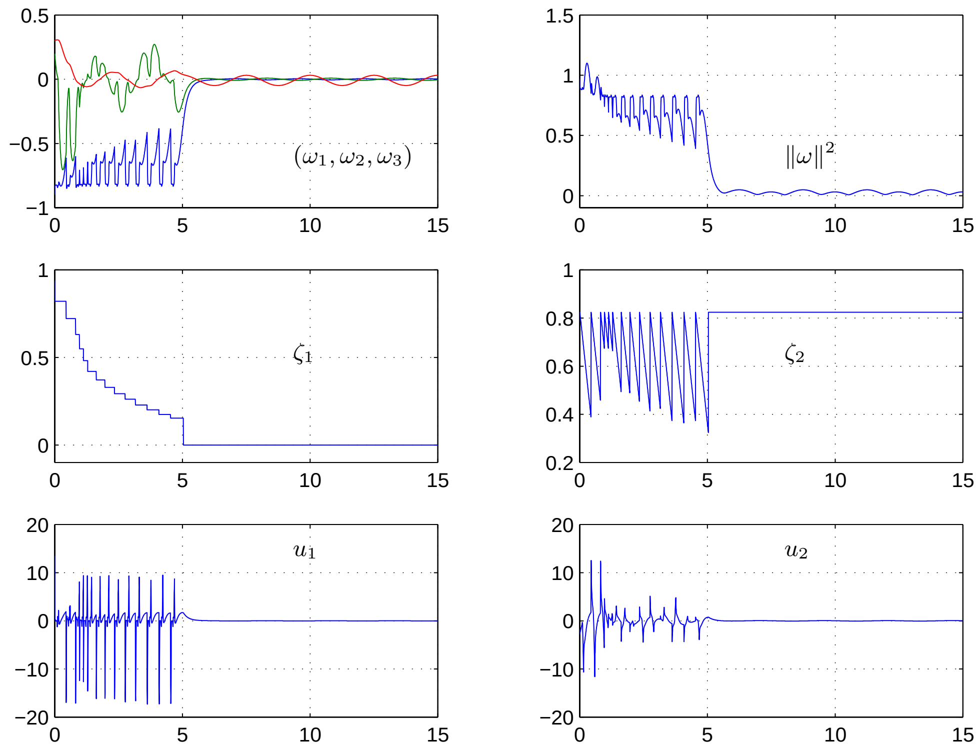


Figure 6: Disturbed case: $b_1 = 0.1a_1$, $b_2 = 0.1a_2$ and $b_3 = 0.1$

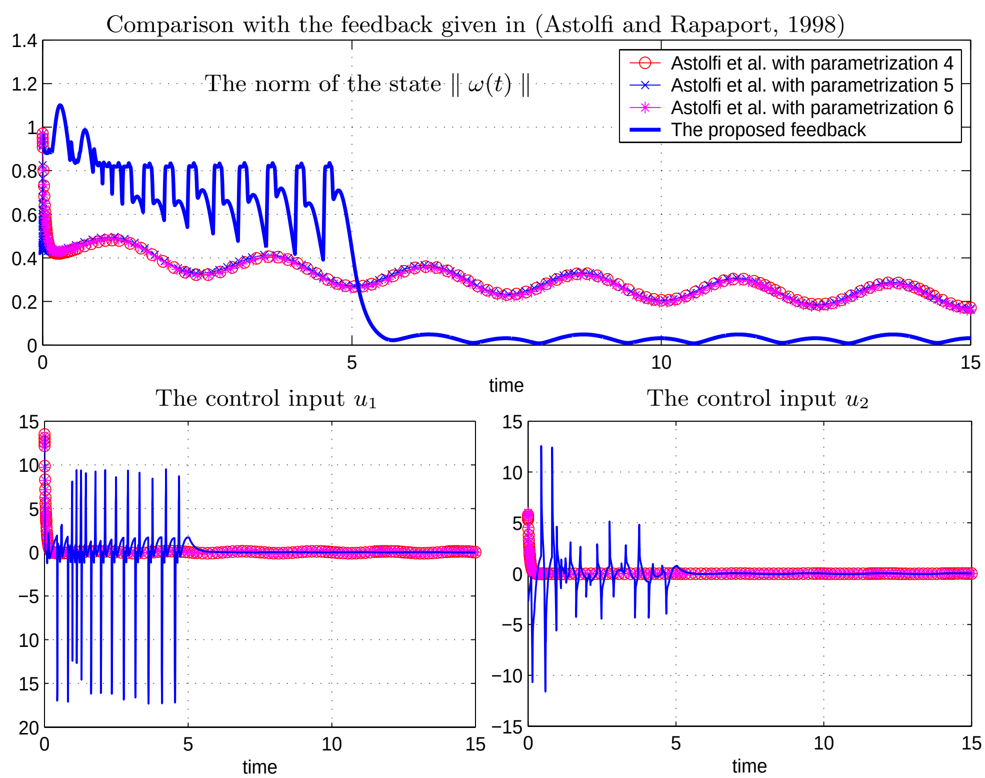


Figure 7: Disturbed case: $b_1 = 0.1a_1$, $b_2 = 0.1a_2$ and $b_3 = 0.1$

to be interesting, this seems to be obtained at the price of quite nervous behaviour of the control. Attempts to attenuate this feature are to be done following the same techniques already used in the sliding mode control literature. On the other hand, the solution of the underlying Hamilton-Jacobi-Isaacs equations is still a tedious task in general. However, the fact that the solution is to be found in a generally short time interval an over a limited "*crown*" of the state space may enables an easier resolution and probably sub-optimal procedure to be found. This constitutes the heart of present investigations.

A Appendix: Proof of theorem 1

- As usual, stability proof is to be done on that uncertain model used to design the feedback law, namely equations (1). Therefore, the only possibly persisting disturbances to be considered are those given by w. Therefore, according to the discussion of section 3, the following implication can be used :

$$\left\{C(x,\zeta)\leq 0\right\}\Rightarrow\left\{\|x\|\leq\gamma\zeta_1\right\} \tag{26}$$

- Let us define the map $\Gamma:\mathbb{R}^+\rightarrow\mathbb{R}^+$as follows :

$$\Gamma(y):=\sup_{\|x\|^2\leq y} V(0,x) \tag{27}$$

- Consider the function $U(x,\zeta)$ defined by :

$$U(x,\zeta):=V(T-\zeta_2,x)+\sum_{k=1}^{k_{max}(\zeta_1)}\Gamma(\gamma^k\zeta_1) \tag{28}$$

  the integer function $k_{max}:\mathbb{R}^+\rightarrow\mathbb{N}$ is given by :

$$k_{max}(\zeta_1)=\min_{k\geq 2}\left\{k\in\mathbb{N}\quad|\quad\Gamma(\gamma^k\zeta_1)\leq\min(\rho_1,\rho_3)\psi(\sqrt{\gamma\zeta_1})\right\} \tag{29}$$

  Note that $k_{max}(\zeta_1)\geq 2$ is well defined since $\Gamma(0)=0$ and $\Gamma(.)$ is continuous.

- The remainder aims to prove that $U(x,\zeta)$ is a somehow suitable Lyapunov function of the stabilization result stated in theorem 1 in the following sense :

  1. $U(x,\zeta)$ is bounded below and proper w.r.t x, that is $\lim_{\|x\|\rightarrow\infty}U(x,\zeta)=\infty$,
  2. At commutation instants t_i such that $\|x(t_i)\|\geq r$, jumps correspond to a decrease in U, namely

$$U(x(t_i^+),\zeta(t_i^+))\leq U(x(t_i),\zeta(t_i)) \tag{30}$$

  3. For all instant t that is not a commutation instant, there is a positive definite function $W(x)$ such that, **on the closed-loop trajectory**, one has:

$$\frac{dU}{dt}(x(t),\zeta(t))\leq -W(x(t))\quad\text{whenever } \|x(t)\|\geq r \tag{31}$$

  4. Starting at some $x(t)$ such that $\|x(t)\|\leq r$, closed-loop trajectories remain in $\mathcal{A}_r$.

  The conjunction of the above facts clearly proves the results.

- **proof of 1**
  This comes from the the fact that $V(T-\zeta_2,x(t))$ is the optimal solution of (6). Indeed, this clearly implies that $U(x,\zeta)\geq V(T-\zeta_2,x)\geq 0$ for all $\zeta_2\in[0,T]$. On the other hand, using the natural assumption 1, it comes that $U(x,\zeta)\geq V(T-\zeta_2)\geq\min(\rho_1,\rho_3)(1+\beta(T))\psi(\|x(t)\|)$ and since $\psi\in\mathcal{K}_\infty$, the result follows.

- **Proof of 2**
Note that at commutation instant t_i, one has

$$x(t_i^+) = x(t_i) \quad (32)$$
$$\zeta_1(t_i^+) = \gamma\zeta_1(t_i) = \|x(t_i)\|^2 \quad (33)$$
$$\zeta_2(t_i^+) = T \quad (34)$$

therefore,

$$\Delta U(t_i) := U(x(t_i^+),\zeta(t_i^+)) - U(x(t_i),\zeta(t_i))$$
$$= V(0,x(t_i)) - V(T-\zeta_2,x(t_i)) + \sum_{k=1}^{k_{max}(\gamma\zeta_1(t_i))} \Gamma(\gamma^{k+1}\zeta_1(t_i)) - \sum_{k=1}^{k_{max}(\zeta_1(t_i))} \Gamma(\gamma^k\zeta_1(t_i)) \quad (35)$$

- using the following property implied by the definition of $k_{max}(.)$:

$$k_{max}(\gamma\zeta_1) = \begin{cases} k_{max}(\zeta_1)-1 & \text{if } k_{max}(\zeta_1(t_i)) > 2 \\ 2 & \text{if } k_{max}(\zeta_1(t_i)) = 2 \end{cases}$$

one can write :

$$\sum_{k=1}^{k_{max}(\gamma\zeta_1(t_i))} \Gamma(\gamma^{k+1}\zeta_1(t_i)) = \begin{cases} \sum_{k=2}^{k_{max}(\zeta_1(t_i))} \Gamma(\gamma^k\zeta_1(t_i)) & \text{if } k_{max}(\zeta_1(t_i)) > 2 \\ \sum_{k=2}^{3} \Gamma(\gamma^k\zeta_1(t_i)) & \text{if } k_{max}(\zeta_1(t_i)) = 2 \end{cases} \quad (36)$$

and using (36) in (35) together with (33) gives :

$$\Delta U(t_i) = \begin{cases} V(0,x(t_i)) - V(T-\zeta_2,x(t_i)) - \Gamma(\|x(t_i)\|^2) & \text{if } k_{max}(\zeta_1(t_i)) > 2 \\ V(0,x(t_i)) - V(T-\zeta_2,x(t_i)) - \Gamma(\|x(t_i)\|^2) + \Gamma(\gamma^3\zeta_1) & \text{if } k_{max}(\zeta_1(t_i)) = 2 \end{cases} \quad (37)$$

and since by definition of $\Gamma(.)$, $\Gamma(\|x(t_i)\|^2) \geq V(0,x(t_i))$, (37) becomes :

$$\Delta U(t_i) = \begin{cases} -V(T-\zeta_2,x(t_i)) & \text{if } k_{max}(\zeta_1(t_i)) > 2 \\ -V(T-\zeta_2,x(t_i)) + \Gamma(\gamma^3\zeta_1) & \text{if } k_{max}(\zeta_1(t_i)) = 2 \end{cases} \quad (38)$$

this ends the proof of 2. in that case $k_{max}(\zeta_1(t_i)) > 2$.

- Now, by the natural assumption 1 and (33), it comes that

$$V(T-\zeta_2(t_i),x(t_i)) \geq \min(\rho_1,\rho_3)\psi(\|x(t_i)\|) \geq \min(\rho_1,\rho_3)\psi(\sqrt{\gamma\zeta_1})$$

this shows that for $k_{max}(\zeta_1) = 2$, one clearly has by definition of $k_{max}(.)$ [see (29)] :

$$-V(T-\zeta_2,x(t_i)) + \Gamma(\gamma^3\zeta_1) \leq 0$$

this with (38) end the proof of 2.

- **Proof of 3.**
Note that if t is not a commutation instant then $\dot{\zeta}_1 = 0$ and the summation term in (28) is constant. Therefore, for all $x(t)$ such that $\|x(t)\| \geq r$, one has $\frac{dU}{dt}(x(t),\zeta(t)) = \frac{dV}{dt}(T-\zeta_2,x(t)) \leq -\rho_3\|x(t)\|^2$ according to conventional arguments from game theory. This proves 3. with $W(x) = \rho_3\|x\|^2$.

- **Proof of 4.**
Since $B_r \subset \mathcal{A}_r$, the only interesting situations are those where trajectories leave B_r. Suppose therefore that $x(t)$ is such that $\|x(t)\| = r$. In this case, (5) holds and using the natural assumption 1, the following inequalities hold :

$$\gamma r^2 \geq V(0,x(t)) \geq \min(\rho_1,\rho_3)(1+\beta(T))\psi(\sup_{t\leq\tau\leq t+T} \|x(\tau)\|) \quad (39)$$

which clearly gives 4. since between t and $t+T$, there is some t_1 for which $x(t_1) \in B_r$, equation (39) shows that the intermediate excursion over $[t,t_1]$ does remain in $\mathcal{A}_r$ ◇

References

Aeyels, D. (1985). Stabilization of a class of nonlinear systems by a smooth feedback control. *Systems & Control Letters* **5**, 289–294.

Aeyels, D. and M. Szafranski (1988). Comments on the stabilizability of the angular velocity of a rigid body. *Systems & Control Letters* **10**, 35–39.

Alamir, M. (2001). Solutions of nonlinear optimal and robust control problems via a mixed collocation/DAE's based algorithm. *Automatica* **37**(7), 1109–1115.

Astolfi, A. and A. Rapaport (1998). Robust stabilization of the angular velocity of a rigid body. *Systems & Control Letters* **34**, 257–264.

Ball, J. A. and J. W. Helton (1992). Nonlinear H_∞ control theory for stable plants. *Mathematics of Control Signals, and Systems* **5**, 233–261.

Ball, J. A., J. W. Helton and M. L. Walker (1993). H_∞ control for nonlinear systems with output feedback. *IEEE Trans. on Autom. Control* **38, No 4**, 546–559.

Balloul, I. (2000). Commande robuste des systèmes non linéaires. Thèse de doctorat. Laboratoire d'Automatique de Grenoble.

Basar, T. and G. J. Olsder (1982). *Dynamic Noncooperative Game Theory*. Academic Press.

Beard, R. W. and T. W. McLain (1998). Successive galerkin approximation algorithms for nonlinear optimal abd robust control. *Int. Journal of Control* **71, No 5**, 717–743.

Beard, R. W., G. N. Saridis and J. T. Wen (1997). Galerkin approximation of the generalized hamilton-jacobi-bellman equation. *Automatica* **33, No 12**, 2159–2177.

Beard, R. W., G. N. Saridis and J. T. Wen (1998). Approximate solutions to the time-invariant hamilton-jacobi-bellman equation. *Journal of Optimization Theory and Applications* **96, No 3**, 589–626.

Brockett, R. W., R. S. Millman and H. S. Sussmann (1983). *Differential Geometric Control Theory*. Chap. Asymptotic stability and feedback stabilization. Birkhauser.

Chen, G. and J. P. Figueiredo (1989). On robust stabilization of nonlinear systems. *Systems & Control Letters* **12**, 373–379.

Clarke, F. H. (1990). *Optimization and Nonsmooth Analysis*. Society for Industrial and Applied Mathematics, Philadelphia.

Cong, L. and P. H. Landers (1995). Robust control of mimo nonlinear systems with unmatched uncertainties. *Int. Journal of Control* **62, No 4**, 961–981.

Georges, D. (1996). Solutions of nonlinear optimal regulator and H_∞ control problems via galerkin methods. *European Journal of Control* **2**, 211–226.

Georgiou, T. T. and M. C. Smith (1997). Robustness analysis of nonlinear feedback systems : An input-output approach. *IEEE Trans. on Autom. Control* **42, No 9**, 1200–1221.

Imura, J., , H. Maeda, T. Sugie and T. Yoshikawa (1995). Robust stabilization of nonlinear systems by H_∞ state feedback. *Systems & Control Letters* **24**, 103–114.

Isidori, A. and A. Astolfi (1992). Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems. *IEEE Transaction on Automatic Control* **37, No 9**, 1283–1293.

Isidori, A. and W. Lin (1998). Global l_2-gain design for a class of nonlinear systems. *Systems & Control Letters* **34**, 295–302.

James, M. R. (1995). Robust H_∞ output feedback control for nonlinear systems. *IEEE Trans. on Autom. Control* **40, No 6**, 1007–1017.

James, M. R. and J. S. Baras (1996). Partially observed differential games, infinite-dimensional hamilton-jacobi-isaacs equations, and nonlinear H^∞ control. *Siam Journal on Control and Optimization* **34, No 4**, 1342–1364.

Li, Z. H., T. Y. Chai and C. Wen (1995). Systematic design of robust controllers for nonlinear uncertain systems. *Int. J. Control* **62, No 4**, 871–892.

Lin, W. (1996). Mixed H_2/H_∞ control via state feedback for nonlinear systems. *Int. J. Control* **64, No 5**, 899–922.

Lu, W. M. and J. C. Doyle (1994). H_∞ control of nonlinear systems via output feedback : Controller parameterization. *IEEE Trans. on Autom. Control* **39, 12**, 2517–2521.

Marchand, N., M. Alamir and I. Balloul (2000). Stabilization of nonlinear systems by discontinuous state feedback. In: *NCN Springer-Verlag series. Lecture Notes in Control and Information Sciences*. Springer-Verlag.

Najson, F. and E. Kreindler (1996). On the lyapunov approach to robust stabilisation of uncertain nonlinear systems. *Int. J. of Robust and Nonlinear Control* **6**, 41–63.

Outbib, R. (1994). On global feedback stabilization of the angular velocity of a rigid body. In: *Proceedings of IEEE conf. on Decision and Control*.

Outbib, R. and G. Sallet (1992). Stabilizability of the angular velocity of a rigid body. *Systems & Control Letters* **18**, 93–98.

Qu, Z. (1992). Global stabilization of nonlinea systems with a class of unmatched uncertainties. *Systems & Control Letters* **18**, 301–307.

Savkin, A. V. and I. R. Petersen (1996). Robust control with a terminal state constraint. *Automatica* **32, No 7**, 1001–1005.

Savkin, A. V. and I. R. Petersen (1998). Robust stabilization of discrete-time uncertain nonlinear systems. *Journal of Optimization Theory and Applications* **96, No 1**, 87–107.

Savkin, A. V. and I. R. Peterson (1994). A connection between H_∞ control and the absolute stabilizability of uncertain systems. *Systems & Control Letters* **23**, 197–203.

Sontag, E. D. and H. J. Sussman (1988). Further comments on the stabilizability of the angular velocity of a rigid body. *Systems & Control Letters* **12**, 213–217.

Teel, A. (1996). On graphs, conic relations, and input-output stability of nonlinear feedback systems. *IEEE Trans. on Autom. Control* **41, No 5**, 702–709.

Van der Schaft, A. J. (1991). On a state space approach to nonlinear H_∞ control. *Systems & Control Letters* **16**, 1–8.