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Brief Paper

Solutions of nonlinear optimal and robust control problems via a mixed collocation/DAE's based algorithm☆

Mazen Alamir*

Laboratoire d'Automatique de Grenoble, UMR 5528 CNRS-INPG-UJF, BP 46, Domaine Universitaire, Saint Martin d'Hères, France

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Abstract

A new algorithm for computing the solutions of nonlinear optimal and robust H_∞ control problems is proposed. The algorithm is based on the use of the collocation method to transform the PDE's into ODE's. Some convergence results are given and several examples are presented.

Keywords: Nonlinear control; Optimal control; Robust control; Numerical solutions

1. Introduction

The formulations of $\mathscr{H}_2$ optimal and $\mathscr{H}_\infty$ robust control problems for nonlinear systems are two conceptually powerful means for the design of nonlinear control laws addressing quite sophisticated performance objectives. Theoretical results in nonlinear $\mathscr{H}_2$ control were developed in the late 1950s and 1960s (Athans & Falb, 1966; Fleming & Rishel, 1975; Lewis, 1986) while nonlinear $\mathscr{H}_\infty$ theory has been developed more recently (Van der Schaft, 1992; Ball, Helton, & Walker, 1993; Isidori & Astolfi, 1992). The existing approaches to solve the resulting Hamilton–Jacobi–Isaacs (HJI) equations (Georges, 1996; Beard, Saridis, & Wen, 1997, 1998; Beard & McLain, 1998) share a technical requirement that is inherently associated to the Galerkin method, namely the need to perform a high number of multi-dimensional (generally n-dimensional) quadratures.

In this paper, it is shown that these quadratures can be avoided if collocation approach is adopted provided that an adequate stabilizing procedure is used to compensate for the lack of precision generally associated to this approach.

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* Corresponding author. Tel.: +33-4-76-82-6326; fax: +33-4-76-82-6388.

E-mail address: mazen.alamir@inpg.fr (M. Alamir).

This paper is organized as follows: First, the nonlinear optimal and/or robust control problem is briefly recalled together with some basic results from the dynamic games theory. In Section 3, some functional approximation related definitions and results are presented as well as a brief presentation of the post stabilization technique's main results. The proposed algorithm is then clearly introduced in Section 4 together with some convergence analysis. Finally, the proposed algorithm is applied to some examples.

2. Problem formulation

Consider nonlinear systems given by

$$\dot{x} = f(x, u, w, t), \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ w \in \mathbb{R}^p, \tag{1}$$

where x is the state, u is the control and w is some unknown exogenous input and/or vector of parameter uncertainties. The control u and the exogenous input w are subject to the following constraints for all $t \geqslant 0$

$$u(t) \in U(x(t)) \subseteq \mathbb{R}^n, \quad w(t) \in W(x(t)) \subseteq \mathbb{R}^p. \tag{2}$$

Most of the control engineering problems amount to solving the following optimization problem:

Find a feedback strategy $u(\cdot, x(\cdot)) \in U(x(\cdot))$ such that

$$u(t, x(t)) := \arg\left\{\min_{u(\cdot)} \max_{w(\cdot)} J_T(x(t), u(\cdot), w(\cdot), t)\right\}$$

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for all $t \geqslant 0$ under (1), (3)

where

$$J_T(x(t), u(\cdot), w(\cdot), t) := \Psi(x(T)) + \int_t^T L(x(\tau), u(\tau), w(\tau), \tau)\, \mathrm{d}\tau, \tag{4}$$

where $L(\cdot)$ is some penalty function expressing the control objectives. In connection with the above formulation, the Hamiltonian H is defined for all $\lambda \in \mathbb{R}^n$ by

$$H(x, \lambda, u, w, t) := \lambda^{\mathrm{T}} f(x, u, w) + L(x, u, w, t). \tag{5}$$

Throughout this paper, the following assumption is supposed to hold:

Assumption 1 (*Existence of saddle points for H*). For all $(x, \lambda, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$, there are $\hat{u}(x, \lambda, t)$ and $\hat{w}(x, \lambda, t)$ solutions of the following static game

$$\hat{H}(x, \lambda, t) := \min_{u \in U(x)} \max_{w \in W(x)} H(x, \lambda, u, w, t). \tag{6}$$

Under the above assumption, Isaacs' "*verification theorem*" may be applied (Isaacs, 1954–1956; Basar & Olsder, 1982).

Theorem 1 (Isaacs verification theorem). If $V(t, x)$ *is a function of class* C^1 *in t and x that satisfies the following Hamilton Jacobi Equation with boundary condition*:

$$V_t + \hat{H}(x, V_x, t) = 0, \quad V(T, x) = \Psi(x), \tag{7}$$

where $\hat{H}$ *is given by* (6), then *the control strategy*

$$u(\tau, x(\tau)) = \hat{u}(x(\tau), V_x(\tau, x(\tau)), \tau) \quad \tau \in [t, T] \tag{8}$$

is an optimal solution for problem (3). *Furthermore, the corresponding optimal value is exactly* $V(t, x(t))$.

Very frequently, one is interested in finding a time independent *nonnegative definite* function $V(x)$, solution of the stationary version of Eq. (7), namely

$$\hat{H}(x, V_x) = 0, \quad V \text{ is nonnegative definite.} \tag{9}$$

The aim of this paper is basically to propose a simple and efficient algorithm to solve Eq. (7) or Eq. (9) together with some analysis regarding the conditions under which it may converge.

3. Background

3.1. Functional approximation related definitions

Let $\mathscr{S} \subset \mathbb{R}^n$ be a compact subset over which Eq. (7) or Eq. (9) are to be approximately solved.

Denote by $(\phi_i)_{i=1}^{\infty}$ a *complete* set of C^1 basis functions $\phi_j : \mathbb{R}^n \to \mathbb{R}$ (Fletcher, 1984). The space of all scalar functions that are linear combinations of $(\phi_j)_{j=1}^N$ will be denoted by $\mathbf{V}_N$. In this paper, the compact subset $\mathscr{S}$ under interest is supposed to contain a set of regular points $(x^i)_{i=1}^N$ w.r.t $(\phi_j)_{j=1}^N$, and this, for all $N \in \mathbb{N}$. In what follows, the following notation is used

$$\Phi_N(x) := (\phi_1(x), \ldots, \phi_N(x))^{\mathrm{T}}. \tag{10}$$

Furthermore, when a set of interpolation points $(x^i)_{i=1}^{n_p}$ (containing a regular subset) is fixed, the $n_p \times N$ interpolation matrix M is defined by

$$M := \begin{pmatrix} \Phi_N^{\mathrm{T}}(x^1) \\ \vdots \\ \Phi_N^{\mathrm{T}}(x^{n_p}) \end{pmatrix} \in \mathbb{R}^{n_p \times N}. \tag{11}$$

Note that since $(x^i)_{i=1}^{n_p}$ contains a regular subset, one has that *rank*$(M) = N$ or, equivalently, that $M^{\mathrm{T}}M$ is invertible. Furthermore, given any function $V(x)$, the least squares approximation of $V(\cdot)$ by elements of $\mathbf{V}_N$ using the set of measures $((x^i), V(x^i))_{i=1}^{n_p}$ is readily given by

$$V_N(x) = [\Phi_N^{\mathrm{T}}(x)[M^{\mathrm{T}}M]^{-1}M^{\mathrm{T}}] \begin{pmatrix} V(x^1) \\ \vdots \\ V(x^{n_p}) \end{pmatrix}. \tag{12}$$

Finally, given a function $F : \mathbb{R}^s \times \mathbb{R}^n \to \mathbb{R}^q : (y, x) \rightsquigarrow F(y, x)$, let us define $\bar{F} : \mathbb{R}^s \to \mathbb{R}^{q \cdot n_p}$ as follows

$$\bar{F}(y) = (F^{\mathrm{T}}(y, x^1), \ldots, F^{\mathrm{T}}(y, x^{n_p}))^{\mathrm{T}} \in \mathbb{R}^{q \cdot n_p}, \tag{13}$$

this notation will be extensively used in the formulation of the proposed algorithm.

3.2. Review of stabilization of ODEs with invariant sub-manifolds

Consider a nonlinear differential system given by

$$\dot{z} = f(z), \quad f \text{ continuous} \tag{14}$$

and suppose that, theoretically, the above differential system admits the following invariant sub-manifold

$$\mathscr{M} := \{z \mid Lz = 0\}, \tag{15}$$

where L is some matrix of convenient dimensions. Eqs. (14) and (15) can be viewed as an overdetermined system of DAEs satisfied by the exact solution for all initial conditions $z_0 \in \mathscr{M}$. Once the system is discretized in order to solve it numerically, however, the full overdetermined system may not have a solution anymore. The question becomes how to design an integration method that explicitly takes into account the above invariance property to yield a stable and consistent integration process.

Proposition 1 (Continuous stabilization, Ascher, 1997). *Consider the ODE* (14). *Apply the stabilization*

$$\dot{z} = f(z) - \alpha K z, \tag{16}$$

where $K = L^{\mathrm{T}}Q$ *and* Q *is a solution of the linear system* $(LL^{\mathrm{T}})Q = L$. *Then*, if *there is some* $\alpha_0 \geqslant 0$ *such that*

$$\|Lf(z)\| \leqslant \alpha_0 \|Lz\| \quad \text{locally near } \mathscr{M}, \tag{17}$$

then, *under the dynamics* (16), $\mathscr{M}$ *is a locally asymptotically stable invariant manifold for* (16) *whenever* $\alpha > \alpha_0$.

The stabilized system (16) must now be discretized in order to obtain a numerical solution. It is a known fact that this discretization process may destabilize the invariant set $\mathscr{M}$. The so-called "*Post-stabilization*" technique enables the precision and the stability of a given numerical scheme to be increased. Namely, if an approximate solution $z_n := z(t_n)$ of (14) is to be found at sampling instants t_n such that $t_{n+1} = t_n + k$, the following equations are used to generate $(z_n)_{n \geqslant 0}$:

$$\tilde{z}_{n+1} = \theta_k^p(z_n), \tag{18a}$$

$$z_{n+1} = (I - K)\tilde{z}_{n+1}, \tag{18b}$$

where $\theta_k^p(z_n)$ is the numerical solution at $t_{n+1} = t_n + k$ of (16) using a pth order integration scheme and starting at the initial condition $z(t_n) = z_n$.

Suppose that one is interested in finding the solution of (14)–(15) over $[0, T]$ where $T > 0$ is some fixed final time. Suppose, however, that only approximated knowledge of $f(z)$ is possible. More precisely, consider the modified version of Eqs. (18a)–(18b)

$$\tilde{z}^{\varepsilon}_{n+1} = \theta_k^p(z^{\varepsilon}_n, \varepsilon(\cdot)), \tag{19a}$$

$$z^{\varepsilon}_{n+1} = (I - K)\tilde{z}^{\varepsilon}_{n+1}, \tag{19b}$$

where $\theta_k^p(z_n, \varepsilon(\cdot))$ is the solution at instant $t_{n+1} = t_n + k$ with initial condition $z(t_n) = z_n$ of the following differential system

$$\dot{z} = f(z) + \varepsilon(t) \tag{20}$$

and $\varepsilon(\cdot)$ is some error profile affecting the knowledge of f.

Using some straightforward continuity arguments concerning the solutions of ODEs with continuous right-hand side over a finite time horizons (Arnold, 1984), the following result can be easily established:

Proposition 2 (Case of perturbed dynamics). If *the nominal system* (14)–(15) *admits a solution on* $[0, T]$, then *there is sufficiently small* $\varepsilon_0 > 0$ *such that the perturbed system* (20) *admits a solution on* $[0, T]$ *for all* $\varepsilon(\cdot)$ *satisfying* $\|\varepsilon\|_\infty^{[0,T]} \leqslant \varepsilon_0$.

Furthermore, *for all n such that* $t_n \in [0, T]$,

$$\lim_{\|\varepsilon\|_\infty^{[0,T]} \to 0} \|z^{\varepsilon}_n - z_n\| = 0. \tag{21}$$

4. Main results

4.1. The proposed algorithm

In order to progressively introduce the algorithm, it will be *temporarily* assumed that the solution $V(t, x)$ of (7) is such that $V(t, .)$ belongs to $\mathbf{V}_N$ for all t.

The above assumption on $V(t, x)$ $[V(t,.) \in \mathbf{V}_N]$ together with the regularity of $(x^i)_{i=1}^{n_p}$ enables to write for all t and all x:

$$V(t, x) = [\Phi_N^{\mathrm{T}}(x)[M^{\mathrm{T}}M]^{-1}M^{\mathrm{T}}]\bar{V}(t), \tag{22}$$

where $M \in \mathbb{R}^{n_p \times N}$ is given by (11) and $\bar{V}(t)$ is the function associated with $V(t, x)$ according to (13). From (22), it is possible to express $V_x(t, x)$ in terms of $\bar{V}(t)$, let $V_x(t, x) =: G(\bar{V}(t), x)$ and define $\Gamma(t, \bar{V}, x)$ as follows:

$$\Gamma(t, \bar{V}, x) := -\hat{H}(x, G(\bar{V}, x), t), \tag{23}$$

then (7) can be used to derive the dynamic equation for $\bar{V}$ together with the associated final condition

$$\frac{\mathrm{d}\bar{V}}{\mathrm{d}t} = \bar{\Gamma}(t, \bar{V}), \quad \bar{V}(T) = \bar{\Psi}, \tag{24}$$

where, again, $\bar{\Gamma}$ (resp. $\bar{\Psi}$) is derived from $\Gamma(t, \bar{V}, x)$ (resp. $\Psi(x)$) using (13).

The differential system (24) with the associated final condition, when integrated backward in time with an infinite precision and using (22) gives the exact solution $V(t, x)$ we search for. The point is that the high dimension of the differential system (24) and the potentially long period over which integration is to be done may lead to a harmful error propagation phenomena. That is the reason why the following algebraic constraint is added to strength the fact that $V(t,.) \in \mathbf{V}_N$

$$[I_{n_p \times n_p} - M(M^{\mathrm{T}}M)^{-1}M^{\mathrm{T}}]\bar{V} =: L\bar{V} = 0. \tag{25}$$

To sum up, the ODEs (24) is to be integrated under the algebraic constraint (25). To do this, the technique recalled in Section 3.2 is used to solve the DAEs (14)–(15). Keeping in mind the notations of Section 3.2, the following correspondences are to be considered:

$$z \leftrightarrow \bar{V}, \quad f \leftrightarrow \bar{\Gamma}, \quad L \text{ given by (25)}.$$

Based on the above discussion, the following algorithm is proposed:

Algorithm

(1) Choose a complete set of C^1 basis functions $(\phi_i(x))_{i=1}^{\infty}$.
(2) Choose a sufficiently high $N \in \mathbb{N}$.
(3) Choose a fixed set of collocation points $(x^i)_{i=1}^{n_p}$ that contains a regular set w.r.t $(\phi_i(x))_{i=1}^{N}$.
(4) Compute
 (a) M according to (11),
 (b) $\Sigma := M(M^{\mathrm{T}}M)^{-1}M^{\mathrm{T}}$,
 (c) $L = I - \Sigma$ [see (25)],
 (d) $K = L^{\mathrm{T}}Q$ where Q is a solution of the linear system $(LL^{\mathrm{T}})Q = L$ (see Proposition 1).
(5) Choose sufficiently high $\alpha > 0$, a sampling period $k > 0$, a small parameter $\varepsilon > 0$ and $T \in \mathbb{R}_+ \cup \{\infty\}$.
(6) $n = 0$; $\bar{V}_n = \bar{\Psi}$.
(7) Compute $\tilde{\bar{V}}_{n+1}$ solution at k of [see (18a)]

$$\dot{\bar{V}}(\tau) = -\bar{\Gamma}(T-\tau,\bar{V}) - \alpha K\bar{V}(\tau); \quad \bar{V}(0) = \bar{V}_n.$$

(8) $\bar{V}_{n+1} = (I-K)\tilde{\bar{V}}_{n+1}$ [see (18b)].
(9) *If* $(nk \geqslant T)$ OR $(\|\dot{\bar{V}}_n\|_\infty \leqslant \epsilon)$ *Then* STOP *Else* $n = n+1$, GOTO Step 7.
(10) The approximate solution is given by

$$V(t,x) \approx [\Phi_N^{\mathrm{T}}(x)[M^{\mathrm{T}}M]^{-1}M^{\mathrm{T}}]\bar{V}_{n+1}(t). \tag{26}$$

4.2. Some convergence results

In this section, the convergence of the above algorithm is investigated. Let us begin by the ideal case:

Proposition 3. If *there is a solution* $V(t,x)$ *of Eq.* (7) *such that for all* $t \geqslant 0$ *one has* $V(t,.) \in \mathbf{V}_N$ then *the differential system* (24) *together with* (22) *admits an exact solution of* (7). Furthermore, $L\bar{V} = 0$ *is an invariant sub-manifold w.r.t the dynamic* $\bar{\Gamma}(t,\bar{V})$ *and hence* Steps 7 *and* 8 *of the above algorithm implement the post stabilization technique on* (24) *yielding a high precision approximation of* $V(t,x)$ *with low computation cost for high* α*'s* whenever *there is some* $\alpha_0 \geqslant 0$ *such that* $\|L\bar{\Gamma}(t,\bar{V})\| \leqslant \alpha_0\|L\bar{V}\|$ *locally near* $L\bar{V} = 0$.

In the case where there is a solution $V(t,x)$ that is not entirely contained in $\mathbf{V}_N$, the following lemma is needed:

Lemma 1. Suppose *that there is a continuously differentiable solution* $V(t,x)$ *of* (7) *over* $[0,T]$. *Let* $\mathscr{S} \subset \mathbb{R}^n$ *be some compact subset of interest and denote by* V_N *the best least squares approximation of* V *by an element that lies in* $\mathbf{V}_N$. Then *there is a family of functions* $\delta_N(t,x)$ *satisfying the following conditions*:

(1) $\lim_{N\to\infty}\|\delta_N(t,.)\|_\infty^{\mathscr{S}} = 0$ *for all* $t \in [0,T]$,
(2) $V_N(t,x)$ *is the solution of the modified optimal control problem*

$$\Psi(x(T)) + \int_t^T [L(x,u,w,\tau) - \delta_N(\tau,x)]\,\mathrm{d}\tau. \tag{27}$$

It can then be readily shown that $\bar{V}_N$ is the solution of the following ODEs that is to be compared to (24):

$$\frac{\mathrm{d}\bar{V}}{\mathrm{d}t} = \bar{\Gamma}_N(t,\bar{V}) := \bar{\Gamma}(t,\bar{V}) + \bar{\delta}_N(t), \quad \bar{V}(T) = \bar{\Psi}. \tag{28}$$

Consequently, the sub-manifold $L\bar{V} = 0$ is invariant w.r.t the dynamic defined by $\mathrm{d}\bar{V}/\mathrm{d}t = \bar{\Gamma}_N(t,\bar{V})$. The above discussion enables the statement of the following main convergence result.

Proposition 4 (Main convergence result). Suppose *that there is an exact* C^1 *solution* $V_{\mathrm{ex}}(t,x)$ *of* (7) *on* $[0,T]$. Let $\mathscr{S}$ *be some compact set of interest and denote by* V_N *the best least squares approximation of* V_{ex} *by an element of* $\mathbf{V}_N$. If *there exists* $\alpha_0 > 0$ *such that* $\|L\bar{\Gamma}_N(t,\bar{V})\| \leqslant \alpha_0\|L\bar{V}\|$ *locally near* $L\bar{V} = 0$ then, *the outputs* $(\bar{V}_n)_{n \leqslant T/k}$ *of the proposed algorithm are such that*

$$\sup_{x\in\mathscr{S}, n \leqslant T/k} \|[\Phi_N^{\mathrm{T}}(x)[M^{\mathrm{T}}M]^{-1}M^{\mathrm{T}}]\bar{V}_n - V_{\mathrm{ex}}(t_n,x)\| \to 0$$

as $N \to \infty$.

5. Illustrative examples

Software has been developed using *digital Fortran* environment while runs have been performed on a 150 MHz *PC-Pentium I*. Integration have been performed using *fourth order Runge–Kutta method*.

The tables that show the convergence results present the evolution with time steps of the following three key quantities: The maximum value of V over the collocation points $(x_i)_{i=1}^{n_p}$: $\|\bar{V}\|_\infty$, The maximum absolute value of $\dot{V}$ over the collocation points $(x_i)_{i=1}^{n_p}$: $\|\dot{\bar{V}}\|_\infty$ and The maximum value of the post stabilization related correction $\|K\tilde{\bar{V}}\|_\infty$ performed in *Step 8* of the algorithm.

It is worth noting that in the following simulations a fixed step Runge–Kutta method has been used. This means that the execution times reported here are extremely pessimistic.

5.1. Example 1

5.1.1. Problem statement

As a first example, let us consider the following nonlinear system that has been considered in Beard et al. (1997):

$$\dot{x} = \begin{pmatrix} -x_1^3 - x_2 \\ x_1 + x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

As pointed out in Beard et al. (1997), the above system can be stabilized using the state feedback

$$u(x) = 3x_1^5 + 3x_1^2 x_2 - x_2 + v$$

and using the coordinate transformation

$$z_1 = x_1, \quad z_2 = x_1^3 + x_2, \tag{29}$$

the system can be written in the new coordinate z as follows:

$$\dot{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

so that the optimal solution of the LQR problem associated to the cost function

$$J = \int_0^\infty [z^T z + v^2]\,dt$$

is given by $z^T P z$, where P is the solution of LQR problem defined by the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$Q = I_{2\times 2}$ and $R = 1$. Calculations show that

$$V(x) = 1.9123x_1^2 - 0.8284x_1x_2 + 1.3522x_2^2 - 0.8284x_1^4 + 2.7044x_1^3x_2 + 1.3522x_1^6. \tag{30}$$

In order to test the proposed algorithm, *the coordinate transformation* (29) *is hidden* and the following nonlinear optimal control problem is defined:

$$\dot{x} = \begin{pmatrix} -x_1^3 - x_2 \\ x_1 + 3x_1^5 + 3x_1^2x_2 + v \end{pmatrix},$$

$$J = \int_0^\infty [x_1^2 + (x_1^3 + x_2)^2 + v^2]\,dt \tag{31}$$

which is in the form (4) with $\Psi \equiv 0$, $T = \infty$ and $L(x, u, w) = x_1^2 + (x_1^3 + x_2)^2 + u^2$. Furthermore, the system being affine in control and the cost function quadratic, assumption 1 is clearly satisfied.

5.1.2. Results

The following functional basis has been considered

$$(\phi_i)_{i=1}^{28} := \{1, x_1, x_2, x_1^2, x_1x_2, x_2^2, x_1^3, \ldots, x_2^6\}$$

furthermore, the parameters have been taken as follows:

$$n_p = 56, \quad \varepsilon = 0.01, \quad \alpha = 5, \quad k = 0.05,$$

the interpolation points have been randomly chosen in the hyper cube $[-1,1] \times [-1,1]$.

Table 1
Convergence result for Example 5.1 (execution time ≈ 5 s)

$t = nk$	$\Vert\bar{V}\Vert_\infty$	$\Vert\dot{\bar{V}}\Vert_\infty$	$\Vert K\tilde{\bar{V}}\Vert_\infty$
0.25	0.46	1.79	8.61E-07
1.00	1.49	0.88	7.70E-06
2.00	1.94	0.37	5.65E-06
3.00	2.08	0.10	2.45E-06
4.00	2.12	2.18E-02	8.40E-07
4.75	2.13	1.06E-02	5.29E-07

Table 2
Convergence results for Example 5.2 (execution time ≈ 25 s)

$t = nk$	$\Vert\bar{V}\Vert_\infty$	$\Vert\dot{\bar{V}}\Vert_\infty$	$\Vert K\tilde{\bar{V}}\Vert_\infty$
0.1	4.35E-02	0.347	2.113E-05
1.0	0.276	0.155	1.665E-05
2.0	0.351	2.666E-02	1.490E-05
3.0	0.362	4.832E-03	1.509E-05
3.5	0.363	1.946E-03	1.515E-05

Convergence results are shown on Table 1.

Neglecting all coefficients that are smaller than 10^{-3}, one obtains

$$V_{28}(x) = 1.9067x_1^2 - 0.8223x_1x_2 + 1.3480x_2^2 - 0.8223x_1^4 + 2.6961x_1^3x_2 + 1.3480x_1^6$$

which is to be compared to (30).

5.2. Example 2

Consider the underwater vehicle considered in Beard and McLain (1998) to which disturbances have been added:

$$\dot{x} = \begin{pmatrix} -x_1|x_1| + x_3 - x_1|x_1|w_2 + w_1 \\ x_1 \\ -5x_3 + 50u \end{pmatrix}. \tag{32}$$

The aim is to solve a nonlinear H_∞ disturbance attenuation problem. For, we seek for a stationary solution to the min max optimization problem for $\gamma > 0$ as small as possible

$$\min_u \max_w J = \int_0^\infty [\Vert x(\tau)\Vert^2 + 0.01u^2(\tau) - \gamma^2 \Vert w(\tau)\Vert^2]\,d\tau.$$

The above problem has been successfully solved over the cube $[-0.6, 0.6]^3$ and $\gamma = 0.1$ using a quadratic polynomial in the unknown x_1, x_2 and x_3 and taking the

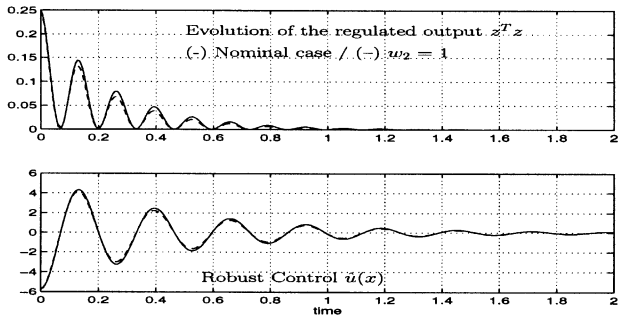

Fig. 1. Closed loop behaviour under parameter uncertainties for Example 5.2.

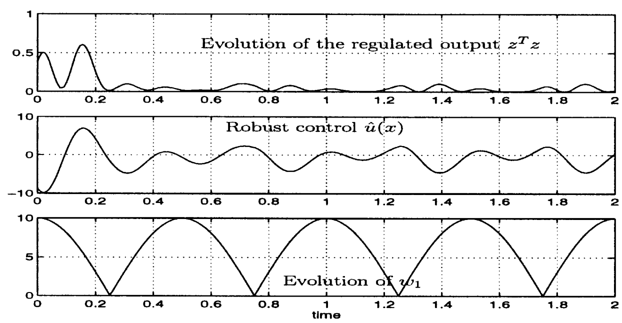

Fig. 2. Closed loop behaviour under external disturbances w_1 for Example 5.2—initial state $x_0 = (0.5, 0.15, -0.6)^T$.

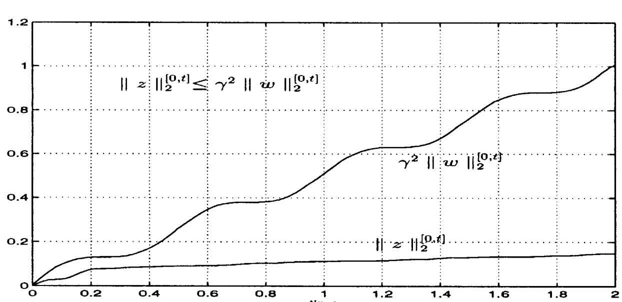

Fig. 3. Test of the H_∞ attenuation inequality for Example 5.2—initial state $x_0 = (0.5, 0.15, -0.6)^T$.

following parameters:

$$n_p = 30, \quad \varepsilon = 0.001, \quad \alpha = 1, \quad k = 0.001$$

to yield the following approximate solution over the hyper cube $[-0.6, 0.6]^3$:

$$V(x) \approx 0.0496x_1^2 + 0.0983x_1x_2 + 0.0046x_1x_3 + 1.0473x_2^2 + 0.0046x_2x_3.$$

Convergence results are presented on Table 2.

5.2.1. Simulation results

Fig. 1 shows the closed loop behaviour in both the nominal case and in the case where $w_2 = 1$. Initial state was taken equal to $x_0 = (0.5, 0.15, -0.6)^T$.

Fig. 2 shows the closed loop behaviour under the external disturbance w_1 for the same initial state. As one can expect with regard to the small value of the attenuation factor $\gamma = 0.1$, the closed loop system presents a very good disturbance rejection behaviour. This is strengthened in Fig. 3 where comparison is made between $\int_0^t z^T z \, d\tau$ and $\gamma^2 \int_0^t w^T w \, d\tau$ attesting that the H_∞ basic inequality is respected.

6. Conclusion

In this paper, an algorithm that solves the Hamilton–Jacobi–Isaacs equations associated to nonlinear optimal and robust control problems is proposed. The algorithm is based on collocation method. The low precision generally associated with this method is compensated by the addition of an algebraic constraint enabling the post-stabilization technique to be used. The proposed algorithm do not remove the basic drawback of the HJIs based approaches, namely, that they become practically unfeasible when the dimension of the state increases. This restricts the domain of application of the proposed algorithm to small systems or to small sub-systems in a system-decomposition-based approaches.

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M. Alamir was born in 1966 at Cairo, Egypt. He Graduated in Mechanical Engineering from the Institut National Polytechnique de Grenoble (INPG), France (1990) and in Aeronautics from the Ecole National, Supérieure des Ingénieurs en Construction Aéronautiques (ENSICA), France (1992). He received a Ph.D. from INPG in Nonlinear Control (1995). Since 1996, he is a research associate CNRS in the Laboratoire d'Automatique de Grenoble. His main research interests are nonlinear receding horizon control, nonlinear receding horizon observers, robust nonlinear and optimal control as well as industrial control applications.