A low dimensional contractive NMPC scheme for nonlinear systems stabilization: Theoretical framework and numerical investigation on relatively fast systems

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Summary. In this paper, a new contractive receding horizon scheme is proposed for the stabilization of constrained nonlinear systems. The proposed formulation uses a free finite prediction horizon without explicit use of a contraction stability constraint. Another appealing feature is the fact that the resulting receding horizon control is in pure feedback form unlike existing contractive schemes where open-loop phases or a memorized threshold are used to ensure the contraction property in closed loop. The control scheme is validating on the swing-up and stabilization problem of a simple and a double inverted pendulums.

1 Introduction

Since the first rigorous proof of the stability of nonlinear receding horizon control schemes [2], it appeared clearly that the closed loop stability is related to some terminal conditions. The early versions of this terminal constraint took the form of an infinite prediction horizon [2] or an equality constraint on the state [2, 3, 4]. These two forms show evident drawbacks since infinite horizon formulations are impossible to compute for general nonlinear systems while the equality constraints on the state makes the underlying optimization problem hardly tractable numerically. These drawbacks gave rise to formulations where the final state is forced to belong to some TERMINAL REGION of appropriate properties. By doing so, the final equality constrained is replaced by an inequality constraint [5, 9, 6]. It goes without saying that an exhaustive survey of all existing formulations that lead to closed loop stability is beyond the scope of the present paper. An excellent survey can be found in [7].

In this paper, interest is focused on contractive receding horizon schemes [8]. These schemes are based on the assumption according to which there exists a contraction factor $\gamma \in [0, 1[$ such that for any initial state x_0 there is a control profile $u(\cdot)$ such that the solution $x_u(\cdot)$ satisfies the contraction constraint $\|x_u(T(x_0))\|_S \leq \gamma \|x_0\|_S$ for some time $T(x_0)$ and some weighting positive definite matrix S. Therefore, given the present state x, the associated open loop optimal control problem is given by [7, 8] :

$$\min_{u(\cdot),T} V(x_u(\cdot,x),T) \quad \text{under} \quad u(\cdot)\in\mathbb{U} \quad \text{and} \quad \|x_u(T,x)\|_S \le \gamma\|x\|_S. \tag{1}$$

Once optimal solutions $\hat{u}(\cdot,x)$ and $\hat{T}(x)$ are obtained, two possible implementations are classically proposed [8]:

- ✓ Either the optimal control $\hat{u}(\cdot,x(t))$ is applied in an open-loop way during the interval $[t,t+\hat{T}(x(t))]$. This means that no feedback is applied during $\hat{T}(x(t))$ time units that may be too long.
- ✓ Or the state $x(t)$ is memorized together with the duration $\hat{T}(x(t))$ and during the interval $[t,\hat{T}(x(t))]$, a sampling period $\tau>0$ is used such that $N\tau=\hat{T}(x(t))$ and a fixed final time receding horizon scheme is used on $[t,t+\hat{T}(x(t))]$ based on the following optimization problem

$$\min_{u(\cdot)} V(x_u(\cdot,x(t+j\tau))) \qquad \text{under} \quad u(\cdot)\in\mathbb{U}$$
$$\text{and} \quad \|x_u(t+\hat{T}(x(t)),x(t+j\tau))\|_S \le \gamma\|x(t)\|_S, \tag{2}$$

which makes the behavior heavily dependent on the past information $x(t)$ and $\hat{T}(x(t))$ that might become irrelevant due to external disturbances that may even make (2) unfeasible. The aim of the present paper is to propose a contractive scheme that leads to a pure state feedback form without memory effect. This is done using the supremum norm and without an explicit contractive constraint in the problem formulation. Furthermore, the open loop control parametrization is explicitly handled by introducing the notion of translatable parametrization. The paper is organized as follows : Section 2 states the problem and gives some notations and definitions. The proposed contractive formulation is presented in section 3 with the related stability results. Finally section 4 shows some illustrative examples.

2 Definitions, notations and problem statement

Consider the class of nonlinear systems given by

$$\dot{x}=f(x,u) \quad ; \quad x\in\mathbb{R}^n \quad ; \quad u\in\mathbb{R}^m, \tag{3}$$

where x and u stand for the state and the control vectors respectively. $F(t,x_0,\mathbf{u})$ denotes the solution of (3) with initial state x_0 under the control profile $\mathbf{u}$ defined on $[0,t]$. The aim of this paper is to define a sampled state feedback of the form :

$$u(t)=K(x(k\tau_s)) \quad ; \quad \forall t\in[k\tau_s\ (k+1)\tau_s[, \tag{4}$$

that asymptotically stabilizes the equilibrium state $x=0$. The following assumption is needed to establish the main result of this paper :

Assumption 1 *For all finite horizon $T>0$, the following asymptotic property holds :*

$$\lim_{\|x_0\|\to\infty}\Big[\min_{\mathbf{u}\in\mathbb{W}^{[0,T]}} \quad \min_{t\in[0,T]} \|F(t,x_0,\mathbf{u})\|\Big]=\infty \tag{5}$$

for all compact subset $\mathbb{W}\subset\mathbb{R}^m$. (In other words, infinitely fast state excursions need infinite control) ♭

Note that assumption 1 is rather technical since it only excludes systems with finite inverse escape time.

2.1 Piece-wise constant control parametrization

Let some sampling period $\tau_s > 0$ be given. One way to define a low dimensional parametrization of piece-wise constant control profiles over the time interval $[0, N\tau_s]$ that belongs to a closed subset $\mathbb{U} \subset \mathbb{R}^m$ is to follow the following two step procedure :

1. First, define a map
$$C: \ \mathbb{P} \to \mathbb{R}^m \times \cdots \times \mathbb{R}^m \quad p \rightsquigarrow C(p) = \big(u^1(p), \dots, u^N(p)\big) \quad ; \quad u^i(p) \in \mathbb{R}^m.$$
2. Project $C(p)$ on the admissible subset $\mathbb{U}^N$ using the projection map $P_{\mathbb{U}}$, namely :
$$P_{\mathbb{U}^N} \circ C: \ \mathbb{P} \to \mathbb{U} \times \cdots \times \mathbb{U} \qquad p \rightsquigarrow P_{\mathbb{U}^N} \circ C(p) = \big(P_{\mathbb{U}}(u^1(p)), \dots, P_{\mathbb{U}}(u^N(p))\big) \ /$$
3. For all $t \in [(k-1)\tau_s, k\tau_s]$, the control is given by $\mathbf{u}(t) = P_{\mathbb{U}}(u^k(p)) =: \mathcal{U}_{pwc}(t, p)$.

Definition 1. *The map C defined above is called the parametrization map while for given C and $\mathbb{U}$, the family $\{\mathcal{U}_{pwc}(\cdot, p)\}_{p\in\mathbb{P}}$ is called a $\mathbb{P}$-admissible parametrization of control profiles.* $\flat$

Definition 2. *A $\mathbb{P}$-admissible parametrization is said to be translatable if and only if for each $p \in \mathbb{P}$, there exists some $p^+ \in \mathbb{P}$ such that $u^i(p^+) = u^{i+1}(p)$ for all $i \in \{1, \dots, N-1\}$* $\flat$

Definition 3. *A $\mathbb{P}$-admissible parametrization $\{\mathcal{U}_{pwc}(\cdot, p)\}_{p\in\mathbb{P}}$ is called* proper *if and only if for all $t_1 < t_2$, one has $\lim_{p\to\infty} \int_{t_1}^{t_2} \|\mathcal{U}_{pwc}(\tau, p)\|^2 \, d\tau = \infty$ whenever $\mathbb{P}$ is radially unbounded.* $\flat$

In what follows, the short notation $F(\cdot, x, p)$ is used instead of $F(\cdot, x, \mathcal{U}_{pwc}(\cdot, p))$.

2.2 The contraction property

Let some sampling period $\tau_s > 0$ be given together with an associated $\mathbb{P}$-admissible control parametrization $\{\mathcal{U}_{pwc}(\cdot, p)\}_{p\in\mathbb{P}}$.

Definition 4. *The system (3) and the control parametrization $\{\mathcal{U}_{pwc}(\cdot, p)\}_{p\in\mathbb{P}}$ satisfy the contraction property if and only if there exists $\gamma \in]0, 1[$ s.t. for all x, there exists $p^c(x) \in \mathbb{P}$ such that :*

$$\min_{q\in\{1,\dots,N\}} \|F(q\tau_s, x, p^c(x))\|^2 \le \gamma \|x\|^2, \tag{6}$$

where $p_c(\cdot)$ is bounded over bounded sets of initial conditions. If moreover, there exists a continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ s.t. for all x :

$$\|F_N(\cdot, x, p^c(x))\|_\infty^2 \le \varphi(x) \cdot \|x\|^2 \quad \text{where} \quad \|F_q(\cdot, x, p)\|_\infty^2 = \max_{i\in\{1,\dots,q\}} \|F(i\tau_s, x, p)\|^2,$$

then the contraction property is said to be strong. $\flat$

2.3 Further notations

For any bounded subset $\mathcal{S}$ of an euclidian space, $\rho(\mathcal{S})$ denotes the radius of $\mathcal{S}$. For all integer $k \in \mathbb{N}$, the notation $k^+ := k+1$ is used. $B(0,r)$ denotes the open ball centered at 0 and of radius r in some euclidian space that is identified from the context. Finally, the projection step is systematically implicitly assumed by writing $u^i(p)$ to denote $P_{\mathbb{U}}(u^i(p))$.

3 A contractive receding-horizon scheme

In all meaningful and realistic applications, there always exists a set of admissible initial conditions, say $\mathbb{X} \subset \mathbb{R}^n$ that corresponds to realistic initial configurations of the controlled system. Therefore, let such subset $\mathbb{X} \subset \mathbb{R}^n$ be fixed once and for all. Assume that a $\mathbb{P}$-admissible control parametrization is defined and that the strong contraction assumption holds (see definition 4). Associated to the set $\mathbb{X}$ of initial conditions, a subset of admissible control parameters, denoted hereafter by $\mathbb{P}_{\mathbb{X}}$ is defined as follows :

$$\mathbb{P}_{\mathbb{X}} := \mathbb{P} \cap B\Big(0, \sup_{x\in \bar{B}(0,\rho(\mathbb{X}))} \|p^c(x)\| + \varepsilon\Big) \subseteq \mathbb{P} \subseteq \mathbb{R}^{n_p}. \tag{7}$$

Namely, a subset of the ball in $\mathbb{R}^{n_p}$ that contains, among others, all the vectors of parameters :

$$\Big\{p^c(x)\Big\}_{x\in \bar{B}(0,\rho(\mathbb{X}))}$$

invoked in the strong contraction assumption. It goes without saying that since $p^c(\cdot)$ is assumed to exist but is not explicitly known, the exact computation of the radius of the ball defining $\mathbb{P}_{\mathbb{X}}$ cannot be easily done. Therefore, in the forthcoming developments, when $\mathbb{P}_{\mathbb{X}}$ is referred to, it is an superset of it that is to be understood. This superset is obtained by taking a sufficiently high radius for a ball in $\mathbb{R}^{n_p}$ centered at the origin.

Consider the following open-loop optimal control problem defined for some $\alpha > 0$ and $\varepsilon > 0$:

$$P_\alpha^{\varepsilon,*}(x) \quad : \quad \min_{(q,p)\in\{1,\dots,N\}\times\mathbb{P}_{\mathbb{X}}} \quad J^*(x,q,p) = \|F(q\tau_s,x,p)\|^2 + \alpha\frac{q}{N}\cdot\min\big\{\varepsilon^2, \|F_q(\cdot,x,p)\|_\infty^2\big\}. \tag{8}$$

Note that if all the functions involved in the definition of the problem (the system's map f and the control parametrization) are continuous then the cost function is continuous in p. This together with the compactness of the set $\mathbb{P}_{\mathbb{X}}$ guarantee that the problem $P_\alpha^{\varepsilon,*}(x)$ admits a solution for all $x \in \mathbb{X}$ and hence is well posed. Therefore, let us denote the solution of (8) for some $x \in \mathbb{X}$ by $\hat{q}(x) \in \{1,\dots,N\}$ and $\hat{p}(x) \in \mathbb{P}_{\mathbb{X}}$. These solutions are then used to define the receding horizon state feedback given by :

$$u(k\tau_s + \tau) = u^1(\hat{p}(x(k\tau_s))) \quad \forall \tau \in [0,\tau_s[. \tag{9}$$

The stability result associated to the resulting feedback strategy is stated in the following proposition :

Proposition 1. If *the following conditions hold :*

1. *The function f in (3) and the parametrization map are continuous and satisfy the strong contraction property (see definition 4). Moreover, the system (3) satisfies assumption 1.*
2. *For all $x \in \mathbb{X}$ and all admissible $\mathbf{u} = \mathcal{U}_{pwc}(\cdot, p)$, the solution of (3) is defined for all $t \in [0, N\tau_s]$ and all $p \in \mathbb{P}_{\mathbb{X}}$. (No explosion in finite time shorter than $N\tau_s$).*
3. *The control parametrization is translatable on $\mathbb{P}_{\mathbb{X}}$ in the sense of definition 2.*

Then, *there exist sufficiently small $\varepsilon > 0$ and $\alpha > 0$ such that the receding horizon state feedback (9) associated to the open-loop optimal control problem (8) is well defined and makes the origin $x = 0$ asymptotically stable for the resulting closed loop dynamics with a region of attraction that contains $\mathbb{X}$.* ♭

Proof

The fact that the feedback law is well defined directly results from the continuity of the functions being involved together with the compactness of $\mathbb{P}_{\mathbb{X}}$. Let us denote by $x_{cl}(\cdot)$ the closed loop trajectory under the receding horizon state feedback law. Let us denote by $V(x)$ the optimal value of the cost function, namely : $V(x) = J^*(x, \hat{q}(x), \hat{p}(x))$.

♭ <u>V is continuous</u>

$V(x)$ can clearly be written as follows

$$V(x) = \inf\Big\{V_1(x), \dots, V_N(x)\Big\} \quad ; \quad V_q(x) := \min_{p \in \mathbb{P}_{\mathbb{X}}} J^*(x, q, p). \tag{10}$$

But for given q, $J^*(x, q, p)$ is continuous in (x, p), therefore $V_q(\cdot)$ is a continuous function of x. Since V is the sum of N continuous functions $(V_j)_{j=1,\dots,N}$, it is continuous itself.

♭ <u>V is radially unbounded</u>

Since the control parametrization is supposed to be continuous, the set of controls given by :

$$\mathbb{U} := \Big\{\mathcal{U}_{pwc}(t, p)\Big\}_{(t,p) \in [0, N\tau_s] \times \mathbb{P}_{\mathbb{X}}},$$

is necessarily bounded. using assumption 1 with $\mathbb{W} = \mathbb{U}$ gives the results.

♭ Finally it is clear that $V(0) = 0$ since zero is an autonomous equilibrium state.

<u>Decreasing properties of V</u>

Two situations have to be distinguished :

Case where $\hat{\mathbf{q}}(\mathbf{k}) > 1$

In this case, let us investigate candidate solutions for the optimization problem $P_\alpha^{\varepsilon,*}(x_{cl}(k^+))$ where $x_{cl}(k^+)$ is the next state on the closed loop trajectory, namely :

$$x_{cl}(k^+) = F\Big(\tau_s, x_{cl}(k), u^1(\hat{p}(x_{cl}(k)))\Big).$$

A natural candidate solution to the optimal control problem $P_\alpha^{\varepsilon,*}(x_{cl}(k^+))$ is the one associated to the translatable character of the control parametrization, namely

$$p_{cand}(k^+) := \hat{p}^+(x_{cl}(k)) \quad ; \quad q_{cand}(k^+) := \hat{q}(x_{cl}(k)) - 1 \geq 1. \tag{11}$$

In the following sequel, the following short notations are used

$$\hat{p}(k) = \hat{p}(x_{cl}(k\tau_s)) \quad ; \quad \hat{q}(k) = \hat{q}(x_{cl}(k\tau_s)) \quad ; \quad V(k) = V(x_{cl}(k)).$$

By the very definition of p^+, it comes that :

$$\begin{aligned} \|F(q_{cand}(k^+)\tau_s, x_{cl}(k^+), p_{cand}(k^+))\|^2 &= \|F(\hat{q}(k)\tau_s, x_{cl}(k), \hat{p}(k))\|^2 \\ = V(x_{cl}(k)) - \alpha\frac{\hat{q}(k)}{N}\min\{\varepsilon, \|F_{\hat{q}(k)}(\cdot, x_{cl}(k), \hat{p}(k))\|_\infty^2\}, \end{aligned} \tag{12}$$

and since $V(x_{cl}(k^+))$ satisfies by definition, one has :

$$\begin{aligned} V(x_{cl}(k^+)) \le \|F(q_{cand}(k^+)\tau_s, x_{cl}(k^+), p_{cand}(k^+))\|^2 + \\ + \alpha\frac{\hat{q}(k)-1}{N}\min\{\varepsilon, \|F_{\hat{q}(k)-1}(\cdot, x_{cl}(k^+), p_{cand}(k^+))\|_\infty^2\}, \end{aligned}$$

This with (12) gives :

$$\begin{aligned} V(x_{cl}(k^+)) \le V(x_{cl}(k)) - \alpha\frac{\hat{q}(k)}{N}\min\{\varepsilon, \|F_{\hat{q}(k)}(\cdot, x_{cl}(k), \hat{p}(k))\|_\infty^2\} + \\ + \alpha\frac{\hat{q}(k)-1}{N}\min\{\varepsilon, \|F_{\hat{q}(k)-1}(\cdot, x_{cl}(k^+), p_{cand}(k^+))\|_\infty^2\}. \end{aligned} \tag{13}$$

But one clearly has by definition of $p_{cand}(k^+)$:

$$\|F_{\hat{q}(k)-1}(\cdot, x_{cl}(k^+), p_{cand}(k^+))\|_\infty^2 \le \|F_{\hat{q}(k)}(\cdot, x_{cl}(k), \hat{p}(k))\|_\infty^2.$$

Using the last equation in (13) gives

$$V(x_{cl}(k^+)) \le V(x_{cl}(k)) - \frac{\alpha}{N}\min\{\varepsilon, \|F_{\hat{q}(k)}(\cdot, x_{cl}(k), \hat{p}(k))\|_\infty^2\}. \tag{14}$$

Case where $\hat{\mathbf{q}}(\mathbf{k}) = 1$

We shall first prove that each time this situation occurs, one necessarily has :

$$x_{cl}(k^+) \in \bar{B}(0, \rho(\mathbb{X})). \tag{15}$$

Proof of (15) Consider a sequence of instant $0 = t_0 < t_1 < \dots < t_N < \dots$ where for all $i \ge 1$, $t_i = k_i\tau_s$ such that $\hat{q}(k_i) = 1$ for all $i \ge 1$ we shall prove the two following facts :

1. $x_{cl}(k_1^+) \in \bar{B}(0, \rho(\mathbb{X}))$
2. If $x_{cl}(k_i^+) \in \bar{B}(0, \rho(\mathbb{X}))$ then $x_{cl}(k_{i+1}^+) \in \bar{B}(0, \rho(\mathbb{X}))$

If these two facts are proved then by induction, it comes that :

$$\{\hat{q}(k) = 1\} \quad \Rightarrow \quad \{x_{cl}(k^+) \in \bar{B}(0, \rho(\mathbb{X}))\}. \tag{16}$$

To prove 1., note that at $k = 0$, $x_{cl}(0) \in \mathbb{X}$ and therefore, the contraction property can be applied to consider $p^c(x_{cl}(0))$ as a candidate value for the initial optimal control problem $P_\alpha^{\varepsilon,*}(x_{cl}(0))$. Therefore,

$$V(x_{cl}(0)) \le \gamma\|x_{cl}(0)\|^2 + \alpha \cdot \varepsilon. \tag{17}$$

Now during the next steps until k_1 occurs, the result (14) can be used to infer that the function V decreases on the closed loop trajectory. Therefore, one has at instant $k_1\tau_s$:

$$V(x_{cl}(k_1)) \leq \gamma\|x_{cl}(0)\|^2 + \alpha\cdot\varepsilon \quad ; \quad \hat{q}(k_1) = 1. \tag{18}$$

But when $\hat{q}(k_1) = 1$, one has also :

$$\|x_{cl}(k_1^+)\|^2 \leq V(x_{cl}(k_1)) \leq \gamma\|x_{cl}(0)\|^2 + \alpha\cdot\varepsilon, \tag{19}$$

and for sufficiently small α and $\varepsilon > 0$, this leads to $x_{cl}(k_1^+) \in \bar{B}(0,\rho(\mathbb{X}))$ which ends the proof of point 1.

The proof of point 2. follows exactly the same argumentation than the one used above starting from the fact that since $x_{cl}(k_i^+)$ is in $\bar{B}(0,\rho(\mathbb{X}))$, one can rewrite the above demonstration with $x_{cl}(k_i^+)$ playing the role of $x_{cl}(0)$ and $x_{cl}(k_{i+1})$ playing that of $x_{cl}(k_1)$. This clearly gives (16). Consequently, by definition of $\mathbb{P}_{\mathbb{X}}$, there exists some $p^c(x_{cl}(k^+))$ such that

$$V(x_{cl}(k^+)) \leq \gamma\|x_{cl}(k^+)\|^2 + \frac{\alpha}{N}\min\{\varepsilon, \|F_N(\cdot, x_{cl}(k^+), p^c(x_{cl}(k^+)))\|_\infty^2)\}. \tag{20}$$

But according to the strong contraction assumption, one has :

$$\|F_N(\cdot, x_{cl}(k^+), p^c(x_{cl}(k^+)))\|_\infty^2 \leq \varphi(x_{cl}(k^+))\cdot\|x_{cl}(k^+)\|^2.$$

therefore (20) becomes ($\lambda := \sup_{\xi\in\bar{B}(0,\rho(\mathbb{X}))}\Big[\varphi(\xi)\Big]$) :

$$\begin{aligned} V(x_{cl}(k^+)) &\leq \gamma\|x_{cl}(k^+)\|^2 + \frac{\alpha}{N}\min\{\varepsilon, \varphi(x_{cl}(k^+))\cdot\|x_{cl}(k^+)\|^2\}, \\ &\leq \gamma\|x_{cl}(k^+)\|^2 + \frac{\alpha}{N}\min\{\varepsilon, \lambda\cdot\|x_{cl}(k^+)\|^2\}. \end{aligned} \tag{21}$$

On the other hand, since $\hat{q}(k) = 1$ by assumption, one clearly has :

$$\|x_{cl}(k^+)\|^2 \leq V(x_{cl}(k)) - \frac{\alpha}{N}\min\{\varepsilon, \|x_{cl}(k^+)\|^2\} \leq V(x_{cl}(k)). \tag{22}$$

Therefore, using (22) in (21) gives $V(x_{cl}(k^+)) \leq \gamma V(x_{cl}(k)) + \alpha\min\{\varepsilon, \lambda\cdot V(x_{cl}(k))\}$ and one can write $V(x_{cl}(k^+)) \leq (\gamma+\alpha\lambda)V(x_{cl}(k))$ which, for sufficiently small α gives $V(x_{cl}(k^+)) \leq \theta\cdot V(x_{cl}(k))$ for $\theta < 1$. To summarize, it has been shown that the optimal cost function $V(x)$ satisfies the following decreasing properties :

$$V(x_{cl}(k^+)) \leq \begin{cases} V(x_{cl}(k)) - \frac{\alpha}{N}\min\{\varepsilon, \|F_{\hat{q}(k)}(\cdot, x_{cl}(k), \hat{p}(k))\|_\infty^2\} & \text{if } \hat{q}(x_{cl}(k)) > 1 \\ \theta\cdot V(x_{cl}(k)) \quad ; \quad \theta < 1 & \text{if } \hat{q}(x_{cl}(k)) = 1 \end{cases} \tag{23}$$

This clearly shows that the closed loop trajectory converges to the largest invariant set contained in

$$\Big\{x\in\mathbb{R}^n \quad | \quad \|F_{\hat{q}(x)}(\cdot, x, \hat{p}(x))\|_\infty = 0\Big\},$$

which clearly shows that $\lim_{k\to\infty} x_{cl}(k) = 0$ by the very definition of $F_q(\cdot, x, p)$. $\diamondsuit$

Note that proposition 1 shows that the contractive receding horizon feedback may be used alone to asymptotically stabilizes the system. However, in many situations,

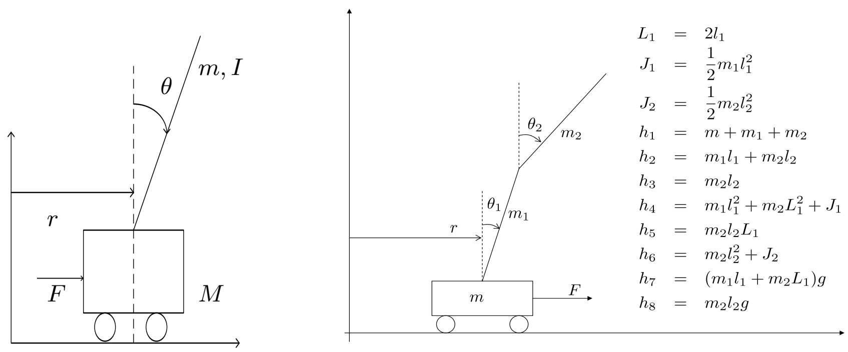

Fig. 1. Description of the simple and the double inverted pendulums

improved behavior around the desired position may be obtained by using the proposed feedback as a steering controller to bring the state to a neighborhood of the desired target and then to switch to some locally stabilizing controller based (for instance) on linearized model. This is commonly referred to as a dual mode control scheme. In the following section, both ways of using the proposed receding horizon feedback are illustrated on two different systems.

Finally, it is worth noting that all the above discussion remains valid if $\|x\|^2$ [resp. $\|F(t,x,p)\|^2$] are replaced by $h(x)$ [resp. $h(F(t,x,p))$] where $h(\cdot)$ is some positive definite function of the state. In this case, the optimization problem (8) writes :

$$P_\alpha^\varepsilon(x) \ : \ \min_{(q,p)\in\{1,\dots,N\}\times\mathbb{P}_\mathbb{X}} J(x,q,p) = h(q\tau_s,x,p) + \alpha\frac{q}{N}\cdot\min\{\varepsilon^2, h_q^\infty(\cdot,x,p)\}, \tag{24}$$

where $h(q\tau_s,x,p) = h(F(q\tau_s,x,p))$ and $h_q^\infty(\cdot,x,p) := \max_{i\in\{1,\dots,N\}} \ h(i\tau_s,x,p)$.

4 Illustrative examples

4.1 Swing-up and stabilization of a simple inverted pendulum on a cart: A stand-alone RHC scheme

The inverted pendulum on a cart is probably the most famous system in the nonlinear control literature (see figure 1). The dynamics of the inverted pendulum can be described by the following equations :

$$\begin{pmatrix} mL^2+I & mL\cos\theta \\ mL\cos\theta & m+M \end{pmatrix}\begin{pmatrix}\ddot\theta \\ \ddot r\end{pmatrix} = \begin{pmatrix} mLg\sin\theta - k_\theta\dot\theta \\ F + mL\dot\theta^2\sin\theta - k_x\dot r\end{pmatrix}. \tag{25}$$

Choosing the state vector $x := \begin{pmatrix}\theta \ r \ \dot\theta \ \dot r\end{pmatrix}^T \in \mathbb{R}^4$ and applying the following precompensation (change in the control variable) :

$$F = -K_{pre} \begin{pmatrix} r \\ \dot{r} \end{pmatrix} + u, \tag{26}$$

where K_{pre} is chosen such that the dynamics $\ddot{r} = -K_{pre} \begin{pmatrix} r \\ \dot{r} \end{pmatrix}$ is asymptotically stable leads to a system of the form (3). Consider the scalar exponential control parametrization (that is clearly translatable with $p^+ = p \cdot e^{-\tau_s/t_r}$) :

$$\mathbb{P} = [p_{min}, p_{max}] \subset \mathbb{R} \quad ; \quad u^i(p) = p \cdot e^{t_i/t_r} \quad ; \quad t_i = \frac{(i-1)\tau_s}{N}, \tag{27}$$

where $\tau_s > 0$ is the control sampling period, $N \cdot \tau_s$ the prediction horizon length while t_r is the characteristic time of the exponential control parametrization.

Define the weighting function $h(x)$ by :

$$h(x) = \frac{1}{2}\left[\dot{\theta}^2 + \beta r^2 + \dot{r}^2\right] + [1 - \cos(\theta)]^2 = \frac{1}{2}\left[x_3^2 + \beta x_2^2 + x_4^2\right] + [1 - \cos(x_1)]^2 \tag{28}$$

In order to explicitly handle the saturation constraint on the force, the constraint has to be expressed in term of the new control variable u, namely :

$$|-K_{pre} \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} + u| \le F_{max}. \tag{29}$$

Using the expression of the control parametrization (27) this yields the following state dependent definition of the parameter bounds p_{min} and p_{max} :

$$p_{min}(x) = -F_{max} + K_{pre_1} x_2 + K_{pre_2} x_4 \tag{30}$$

$$p_{max}(x) = +F_{max} + K_{pre_1} x_2 + K_{pre_2} x_4 \tag{31}$$

These bounds are used in the definition of the optimization problem $P_\alpha^\varepsilon(x)$:

$$P_\alpha^\varepsilon(x) \quad : \quad \min_{(q,p)\in\{1,\dots,N\}\times[p_{min}(x)\ ,\ p_{max}(x)]} J(x,q,p) = h(q\tau_s, x, p) + \frac{\alpha}{N} \cdot \min\{\varepsilon, h_q^\infty(\cdot, x, p)\}. \tag{32}$$

Let $\hat{p}(x)$ and $\hat{q}(x)$ be optimal solutions of $P_\alpha^\varepsilon(x)$. This defines the feedback $K_{RH}(x) = u^1(\hat{p}(x))$ according to the receding horizon principle. The values of the system's parameters used in the forthcoming simulations are given by :

$$(m, M, L, k_x, k_\theta, I) = (0.3, 5.0, 0.3, 0.001, 0.001, 0.009)$$

while the values of the parameters used in the controller definition are the following :

$$(\tau_s, N, t_r, \alpha, \beta) = (0.4, 8, 0.2, 0.01, 10)\ ;\ K_{pre} = (2.5, 10)\ ;\ F_{max} \in \{1, 2\}$$

The behavior of the closed loop systems under the contractive receding horizon control is depicted on figure 2. Two scenarios are presented for different values of the input saturation levels $F_{max} = 1$ and $F_{max} = 2$. The computation times are also given vs the sampling period (the computations have been performed on a 1.3 GHz PC-Pentium III). Note that these computation times never exceeded 0.1 s. This has to be compared to the sampling period $\tau_s = 0.4$ s. This suggests that the proposed receding horizon feedback can be implementable in real time context.

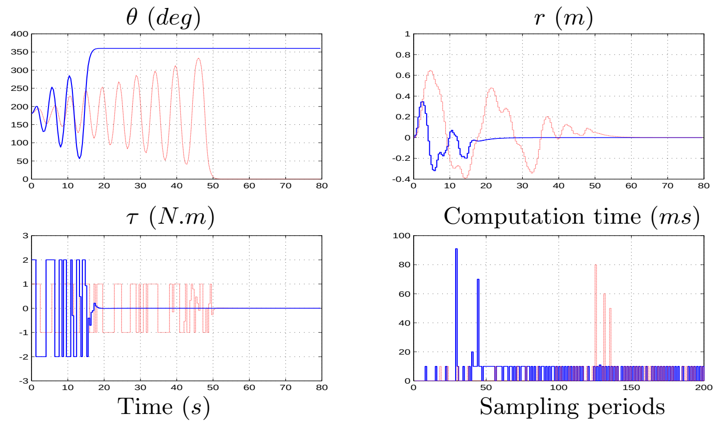

Fig. 2. Stabilization of the inverted pendulum for two different saturation levels: $F_{max} = 1.0\ N$ (dotted thin line) / $F_{max} = 2.0\ N$ (continuous thick line). Initial condition: Downward equilibrium $x = (\pi, 0, 0, 0)^T$.

4.2 Swing up and stabilization of a double inverted pendulum on cart: a hybrid scheme

The system is depicted on figure 1 together with the definition of some auxiliary variables. The numerical values are given by :

$$(m_1, m_2, m, l_1, l_2, J_1, J_2) = (0.3, 0.2, 5.0, 0.3, 0.2, 1.3 \times 10^{-2}, 4 \times 10^{-3}).$$

The system equations are given by [1] :

$$\begin{aligned} h_1\ddot{r} + h_2\ddot{\theta}_1 \cos\theta_1 + h_3\ddot{\theta}_2 \cos\theta_2 &= h_2\dot{\theta}_1^2 \sin\theta_1 + h_3\dot{\theta}_2^2 \sin\theta_2 + F \\ h_2\ddot{r}\cos\theta_1 + h_4\ddot{\theta}_1 + h_5\ddot{\theta}_2 \cos(\theta_1 - \theta_2) &= h_7 \sin\theta_1 - h_5\dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \\ h_3\ddot{r}\cos\theta_2 + h_5\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + h_6\ddot{\theta}_2 &= h_5\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + h_8 \sin\theta_2 \end{aligned}$$

Again, a pre-compensation is done using the change in control variable given by :

$$F = -K_{pre} \cdot \begin{pmatrix} r \\ \dot{r} \end{pmatrix} + u, \tag{33}$$

while a two-dimensional control parametrization is needed this time :

$$\mathbb{P} = [p_{min}, p_{max}]^2 \subset \mathbb{R}^2 \quad ; \quad u^i(p) = p_1 \cdot e^{\lambda_1 t_i} + p_2 e^{-\lambda_2 t_i} \quad ; \quad t_i = \frac{(i-1)\tau_s}{N} \tag{34}$$

The weighting function $h(\cdot)$ invoked in the general formulation (24) is here taken as follows

$$\begin{aligned} h(x) = \frac{h_4}{2}\dot{\theta}_1^2 + \frac{h_6}{2}\dot{\theta}_2^2 + h_5\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) + h_7\big[1 - \cos(\theta_1)\big] + h_8\big[1 - \cos(\theta_2)\big] + \\ + h_1\big[r^2 + \dot{r}^2\big]. \end{aligned}$$

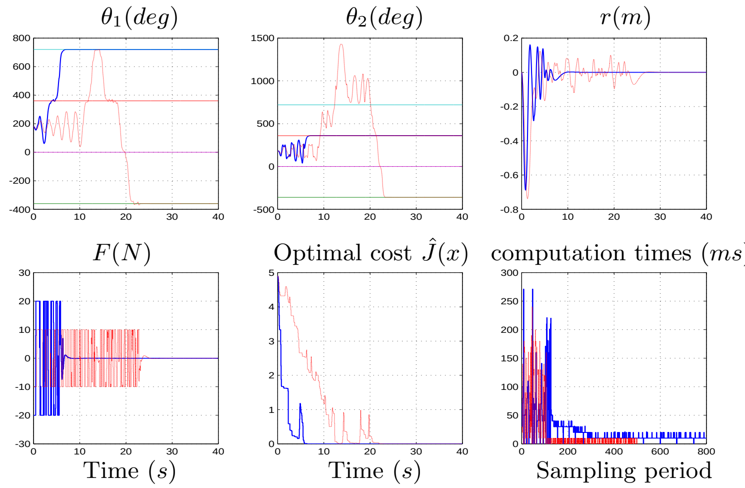

Fig. 3. Closed loop behavior of the double inverted pendulum system under the hybrid controller given by (35) with the design parameters values given by $(\tau_s, N, R, \lambda_1, \lambda_2, \eta) = (0.3, 10, 100, 100, 20, 1)$, $L = (360, 30)$, and $Q = diag(1, 1, 10^4, 1, 1, 1)$ for two different force saturation levels: $F_{max} = 20\ N$ (continuous thick line) / $F_{max} = 10\ N$ (dotted thin line). The maximum number of function evaluations parameter in the optimization code has been set to 20 in order to make the solution real-time implementable. This may explain the behavior of the optimal cost for the lower values that is not monotonically decreasing. Initial condition: downward equilibrium.

This is inspired by the expression of the total energy given in [1]. The constrained open-loop optimal control problem is then given by (24) in which the admissible domain of the parameter vector is $[p_{min}(x), p_{max}(x)]^2$ where :

$$p_{min}(x) := \frac{1}{2}\Big[-F_{max} + K_{pre}\begin{pmatrix} r \\ \dot{r} \end{pmatrix}\Big] \quad ; \quad p_{max}(x) := \frac{1}{2}\Big[+F_{max} + K_{pre}\begin{pmatrix} r \\ \dot{r} \end{pmatrix}\Big].$$

that clearly enables to meet the requirement $|F(t)| \le F_{max}$ given the parametrization (34) being used. Again, denoting by $(\hat{q}(x), \hat{p}(x))$ the optimal solutions, the nonlinear receding-horizon control is given by :

$$u(k\tau_s + t) = K_{RH}(x(k\tau_s)) := u^1(\hat{p}(x(k\tau_s))) \quad ; \quad t \in [0, \tau_s[.$$

Since a hybrid scheme is used here, the local controller has to be defined. This is done by using an LQR-based method that enables a feedback gain L to be computed. Hence, the local controller is given by $K_L(x) = -L \cdot \big(x_1^m\ x_2^m\ x_3\ \dots\ x_6\big)^T$ where x_1^m and x_2^m are the minimum norm angles that are equal (modulo 2π) to θ_1 and θ_2 respectively while the gain matrix $L \in \mathbb{R}^{1\times 6}$ satisfies the following Riccati equation for some positive definite matrices S and Q :

$$A_d^T S A_d - S - (A_d^T S B_d)(R + B_d^T S B_d^T)(B_d^T S A_d) + Q = 0.$$

where (A_d, B_d) are the matrices of the discrete linearized system around the upward position. To summarize, the hybrid controller is given by

$$u(k\tau_s + \tau) = \begin{cases} K_{RH}(x(k\tau_s)) & \text{if} \quad \|x(k\tau_s)\|_S^2 > \eta \\ K_L(x(k\tau_s)) & \text{otherwise} \end{cases} \tag{35}$$

The positive real $\eta > 0$ is a threshold that must be sufficiently small for the ball

$$B_\eta := \left\{ x \in \mathbb{R}^6 \quad \text{s.t} \quad \|x\|_S^2 \leq \eta \right\}$$

to be both entirely included in the region of attraction and invariant under the linear control law $K_L(\cdot)$. Such $\eta > 0$ clearly exists.

The behavior of the closed loop system under the hybrid controller is shown on figure 3 for two different saturation levels $F_{max} = 10\ N$ and $F_{max} = 20\ N$. Note that the maximum number of function evaluations during the on-line optimization has been set to 20. This led to computation times that never exceeded the sampling period $\tau_s = 0.3\ s$.

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