A new Equal-Partition Measurement Encoding Scheme for Networked Control Systems

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Abstract

In this paper, a new encoding strategy is proposed for data exchange through limited bandwidth communication networks. The proposed strategy belongs to the family of *finitely recursive Equal-Partition* schemes. Convergence results are proposed for limited bandwidth encoded signals under bounded measurement noises. When the noise probability density function is known, heuristic procedure is proposed to optimize the encoding scheme's parameters. This is done using the discrete-time Markov chains tool. The whole scheme and the related convergence results are illustrated through concrete simulation examples.

KEYWORDS Networked control systems, encoding, Markov chains, convergence.

1 Introduction

The use of modern communication networks to connect several controlled plants to a distant computation unit is becoming a standard scheme. This networked architecture shows many advantages in comparison with classical point-to-point design. Among others, one may retain their low installation cost, easiness of maintenance and high degree of flexibility [1, 16].

When a dynamical system is controlled through a communication network, data has to be exchanged between the local devices (sensors/actuators) and the distant computation unit. In this context, a trade-off is to be found between the precision level of the exchanged data and the network induced troubles that increase with the data size (delay, data missing, network saturation). In other words, since bandwidth must often be shared across networks, the effective data rate at which sensors can reliably communicate to the computer host may be severely limited, often to a few bits of information per each acquired sensor measurement.

While it is the extensive use of modern communication networks that gave a serious impulse to the field of "*control under limited information*". earlier works should be underlined about encoding in controlled system context [19, 5, 15]. In [19], the trade-off between the sampling rate and the "*wordlength*" used in encoding the measurement is studied in the context of digital Kalman filters. The output measurements are supposed to be normalized and fixed regular quantization is used in the encoding process. In [5], the effect of state quantization is analysed when stabilizing an open-loop unstable linear system while in [15], chaotic behaviours have been proved to exist under some sufficient conditions due to the round-off effects when using finite-wordlength compensators.

Since these earlier works, "*networked control systems*" has became a recognized and very active research field in the control system community [1]. In this paper we focus on the problem of encoding and decoding the measurements sent by a local sensor to a distant computer that performs the state reconstruction in order to compute the corresponding control. Many of the ideas developed in the following sections may be used in the control-input encoding process that can be viewed as the dual process (for control related topics, see for example [21, 16, 17, 18] and the reference therein).

In this paper we are interested in having a "*system estimation*" viewpoint. More precisely, we are interested in transmitting the output measurements of a dynamical system given by

$$\dot{x}(t) = f(x(t), u(t)) + \nu_1(t) \quad (1)$$
$$y_m(t) = h(x(t)) + \nu_2(t) \quad (2)$$

where ν_1 and ν_2 are state and measurement noises respectively. With this respect, the recent works in [10, 20] have to be analyzed in order to clearly state the contribution of the present paper.

In [10], the properties of a "*one bit coder-estimator*" are studied when a linear scalar stochastic process is considered, namely

$$\frac{dX(t)}{dt} = aX(t) \quad ; \quad X(0) = X_0 \quad \text{unknown}$$

where the stochastic variable X_0 is supposed to be $\mathcal{L}(\mu, \lambda)$ (Laplacian). The characteristic sets used are based either on the midpoint partition scheme or on the mean partition scheme. The measurements are supposed to be noise-free. The basic result of [10] is a sufficient condition for the corresponding coder-estimator scheme to converge in quadratic mean. Note that this is possible because no noise is affecting the measurements.

In a more general framework [20], the following stochastic dynamical system is considered

$$\begin{cases} X_{i+1} = f(X_i) + U_i \\ X_0 \quad \text{has a known p.d.f } q \\ Y_{i+1} = X_{i+1} + V_{i+1} \end{cases} \tag{3}$$

where X_i stands for the state, Y_i the measurement while U_i, V_i stand for the state disturbance and the measurement noise respectively.

In [20], two kinds of finitely recursive encoding sequences are investigated

A N-bits Mean Coder-Estimator Sequence This generalizes the framework of [10] where a one-bit scheme is used. The partition sets are still defined using the conditional mean given the past measurements, namely

$$\mathbf{E}\Big[Y(i\delta)|c_0, \dots, c_{i-1}\Big] \tag{4}$$

where $c_0, \dots, c_{i-1}$ are the past N-bits encoding words for $Y(0), \dots, Y((i-1)\delta)$. An upper bound on the estimation error's variance is proposed under some technical assumption on the p.d.f q characterizing X_0.

An Equal-Partition Coder-Estimator Sequence This scheme assumes that the initial state X_0, the state disturbance U_i and the measurement noise V_i satisfy the following boundedness conditions

$$X_0 \in [-x_0, x_0] \quad ; \quad U_i \in [-u, u] \quad ; \quad V_i \in [-v, v]$$

Based on the above assumption and using the system's dynamics (3), the proposed scheme recursively updates the bounds of an interval $[A_i, B_i]$ to which the future measurement certainly belongs, in particular, this involves computations of U_i and L_i such that

$$U_i = \max_{x \in [A_i, B_i]} f(x) + u + v \quad ; \quad L_i = \min_{x \in [A_i, B_i]} f(x) - u - v \tag{5}$$

sufficient conditions are proposed in [20] for the above encoding scheme to be stable in the sense that there is $C < \infty$ such that for all i

$$\mathbf{E}\|\hat{X}_i - X(t_i)\| \leq C \tag{6}$$

where $\hat{X}_i$ is the estimation of $X(t_i)$ based on the past measurements, that is

$$\hat{X}_i := \mathbf{E}\big[X(i\delta)|c_0, \dots, c_i\big]$$

The contribution proposed in the present paper lies on the following limitations of the finitely recursive coder-estimator schemes of [20]

1. The generalization of the encoding scheme of [20] to the general state estimation problem arising in networked control systems where

$$\dot{x} = f(x, u) \quad ; \quad y = h(x) \quad ; \quad x \in \mathbb{R}^n \quad ; \quad y \in \mathbb{R}^p \tag{7}$$

would afford serious difficulties. Indeed, in this case, the computation of (4) for the Mean Coder-Estimator Sequence or (5) for the Equal Partition Coder-Estimator sequence would involve **on-line** solution of multi-dimensional nonlinear optimization problems. For instance, in the context of (7), computing (4) amounts to first estimate the state $\hat{x}(t)$ based on the measurements

$c_0, \dots, c_{i-1}$ and then estimate $\hat{y}(t) = h(\hat{x}(t))$. But it is well known that for systems of the form (7), estimating $\hat{x}(t_i)$ from the knowledge of even free-encoding/free-noise measurements $y(0), \dots, y((i-1)\delta)$ is a quite hard task that underlines the whole still active research field on "*nonlinear state estimation*" leading to the Extended Kalman Filters [3, 13], High-gain observer [4, 9], nonlinear receding-horizon observers [14, 11, 2], etc. Similarly, in the general case (7), solving (5) needs nonlinear constrained programming algorithms to be applied. In the scheme of [20], these computations are to be done at the sensor level.

Solving **on-line** such problems may be incompatible with the local computational facilities at one's disposal at the sensor interface level. It is precisely in order to avoid dispersing the computational capacities that networks are used to share the computation effort of a single computer. The latter may then implement one of the complex state estimation algorithms mentioned above.

2. The second drawback that may call for further improvements is that the encoding schemes of [20] contains no tools to explicitly handle potential specific knowledge about the statistics of U_i and V_i appearing in (3). Only the maximal values of these disturbances/noises are used in the parameterization of the encoder. This is quite harmful in the sense that even when using (3) to write the standard problem of information theory where a static variable is estimated from noisy measurements, one obtains by choosing $f(X) = X_0$ and $U_i \equiv 0$ in (3)

$$Y_{i+1} = Y_0 + V_i$$

and the proposed encoding scheme is the same whatever is the p.d.f of the noise V_i in the support interval $[-v, v]$. This makes the strategy automatically fail to be competitive in comparison to a whole existing literature in a quite simple and frequent situation (see [12] and the references therein).

3. Finally, whatever is the relevance of the framework of the Equal Partition Coder-Estimator sequence proposed in [20], the associated convergence results are quite limited. It seems that this directly results from the fact that the bounding set is designed so that it contains the measured variable at each instant. This requirement may be too strong for interesting results to be derived. This calls for a modified design that may lead to more concrete, computable results and to some optimization rules, probably heuristically defined, making possible an explicit handling of given noise statistics under certain conditions.

In this paper, a modified recursive **E**qual **P**artition N-bits **E**ncoding **S**cheme (**EPES**) is proposed in which the roles of the sensor devices and the distant computer in the overall estimation process are clearly separated. More specifically, the objective of the encoding scheme is to provide the distant computer with measurements that are as close as possible to the real NOISY MEASUREMENTS that it would collect in a network-free context. It is then the role of the state estimation algorithm to retrieve the system state despite the effect of the measurement noise.

In the light of the above discussion, the **EPES** proposed in this paper aims to meet the following requirements:

- Easy extension to the multi-output case where the dimension of the system's state is higher than the number of outputs.
- Low encoding computation cost for the local sensor device
- Explicit computation of encoding scheme's parameters given upper bounds on both the measured signal's bandwidth and the level of measurement noise.
- An explicit rule based on intuitive arguments to account for detailed information about the measurement noise, at least for some simple situations.

As long as stochastic properties are concerned, the basic tool used in the present paper is finite discrete-time Markov chains [8, 6] that provide under certain conditions deep and easily quantifiable insight in the asymptotic properties of stochastic behaviours. It is worth mentioning that Markov chains have been recently used in a control through networks related work [7].

The paper is organized as follows. In section 2, the proposed dynamic encoding scheme is clearly defined. Some convergence results are then derived in section 3 for two different characterizations of the encoded measurement signal. Finally, a heuristic procedure is proposed in section 4 to "*optimize*" the encoding scheme's parameters based on the statistical characterization of the encoded signal. For easiness of reading, all the proofs have been reported in the appendix.

2 Definition of the proposed encoding strategy

Assume that a plant is controlled by a computer through a communication network. To compute the control input to be sent to the local actuator device, the computer performs a dynamic estimation $\hat{x}(t) \in \mathbb{R}^n$ of the plant state $x(t) \in \mathbb{R}^n$ (using available state estimation algorithms) based on past encoded output measurements

$$c(t-\tau)\ ,\ c(t-2\tau), \dots \tag{8}$$

that are short-words versions of the real measurements

$$y_m(t-\tau)\ ,\ y_m(t-2\tau), \dots \tag{9}$$

For simplicity, let us assume that the system is observable with a scalar measurable output. The local sensor device has then to send the measurement to the computer in a coded form in order to reduce data exchanged size. The sensor device and the distant computer are hereafter referred to as "*the partners*".

The encoding strategy is based on the use, at each instant $k\tau$, of an equal partition of the interval

$$\Big[y_c(k-1) - \varepsilon_{min}(k-1), y_c(k-1) + \varepsilon_{max}(k-1)\Big] \tag{10}$$

where

$\mathbf{y_c(k-1)}$ is the decoded value of $y_m(k-1)$ at the preceding instant $(k-1)\tau$. The expression of $y_c(k)$ is given below. This value can be computed by both partners.

$\boldsymbol{\varepsilon_{min}(k)}$ **and** $\boldsymbol{\varepsilon_{max}(k)}$ are varying **positive** parameters with dynamics explicitly given below. Clearly, $\varepsilon_{min}(k)$ and $\varepsilon_{max}(k)$ are coding-encoding parameters that are to be simultaneously updated by both *partners*.

Let us explicit what is exactly done by the transmitter (the sensor device) and the receiver (the distant computer). It is assumed that the wordlength of the exchanged data is fixed to some $N \in \mathbb{N}$ such that $N > 2$.

2.1 Transmitter task

The updating rules for ε_{min} and ε_{max} are given by

$$\varepsilon_{max}(k) = \max\left\{\Psi\Big(\frac{y_m(k)-y_m(k-1)}{\varepsilon_{max}(k-1)}\Big)\cdot\varepsilon_{max}(k-1)\ ,\ \varepsilon_0\right\} \tag{11}$$

$$\varepsilon_{min}(k) = \max\left\{\Psi\Big(\frac{y_m(k-1)-y_m(k)}{\varepsilon_{min}(k-1)}\Big)\cdot\varepsilon_{min}(k-1)\ ,\ \varepsilon_0\right\} \tag{12}$$

where $\varepsilon_0 \geq 0$ is some fixed parameter while $\Psi(.)$ is given by

$$\Psi(\rho) := \left\{\begin{array}{lll}\alpha^{-1} & \text{if} & \rho \leq \alpha^{-1}\\ \alpha & \text{if} & \rho > \alpha^{-1}\end{array}\right. \quad ; \quad \alpha > 1 \tag{13}$$

It is worth noting that the computation of the terms

$$\Psi\Big(\frac{y_m(k)-y_m(k-1)}{\varepsilon_{max}(k-1)}\Big) \quad ; \quad \Psi\Big(\frac{y_m(k-1)-y_m(k)}{\varepsilon_{min}(k-1)}\Big)$$

can be done only by the transmitter since it disposes of the measurements $y_m(k)$ and $y_m(k-1)$. It is therefore necessary to send these terms to the receiver so that identical updating of the equally partitioned encoding interval (10) may be performed by both partners. Fortunately, since α is a shared fixed value, the only necessary information to be sent to the computer is reduced to $i_{max}(k)$ and $i_{min}(k)$ such that

$$\alpha^{i_{max}(k)} = \Psi\Big(\frac{y_m(k)-y_m(k-1)}{\varepsilon_{max}(k-1)}\Big) \quad ; \quad \alpha^{i_{min}(k)} = \Psi\Big(\frac{y_m(k-1)-y_m(k)}{\varepsilon_{min}(k-1)}\Big)$$

and since $\Big(i_{min}(k), i_{max}(k)\Big)$ necessarily belongs to the set

$$\mathcal{I} := \Big\{(1,-1)\ ,\ (-1,-1)\ ,\ (-1,1)\Big\} \tag{14}$$

it comes that the information $\big(i_{min}(k), i_{max}(k)\big)$ needs 2-bits to be encoded.

In addition, the transmitter computes the following encoded value for $y_m(k)$ using the remaining $(N-2)$-bits

$$c(k) = \text{Arg} \min_{i\in\{0,\dots,2^{N-2}-1\}} \Bigg[y_m(k) - y_c(k-1) + \varepsilon_{min}(k-1) \tag{15}$$

$$- \frac{i}{2^{N-2}-1}[\varepsilon_{min}(k-1) + \varepsilon_{max}(k-1)] \Bigg]^2 \tag{16}$$

where $y_c(k-1)$ is the decoded value of the past measurement $y_m(k-1)$ given by (19) in which k is replaced by $k-1$. Note that (16) is clearly the closer grid-point to $y_m(k)$ when $(N-2)$-bits equal partition of the interval

$$\big[y_c(k-1) - \varepsilon_{min}(k-1)\ ,\ y_c(k-1) + \varepsilon_{max}(k-1)\big]$$

is used.

To summarize, the transmitter sends the following two information

- The 2-bits information $\big(i_{min}(k), i_{max}(k)\big)$ necessary to compute the updated values $(\varepsilon_{min}(k), \varepsilon_{max}(k))$.
- The $(N-2)$-bits encoded value $c(k)$ of $y_m(k)$. Note that the computation of (15) do not need high level optimization since it can be easily performed by enumeration over $(N-2)$-values set.

It also computes

- The values of $\varepsilon_{min}(k)$ and $\varepsilon_{max}(k)$ using the following updating rules

$$\varepsilon_{max}(k) = \alpha^{i_{max}(k)} \cdot \varepsilon_{max}(k-1) \tag{17}$$

$$\varepsilon_{min}(k) = \alpha^{i_{min}(k)} \cdot \varepsilon_{min}(k-1) \tag{18}$$

- The encoded value $y_c(k)$ of $y_m(k)$ according to

$$y_c(k) = y_c(k-1) - \varepsilon_{min}(k-1) + \frac{c(k)}{2^{N-2}-1}\Big[\varepsilon_{min}(k-1) + \varepsilon_{max}(k-1)\Big] \tag{19}$$

2.2 Receiver tasks

After reception of the 2-bits information $\big(i_{min}(k), i_{max}(k)\big)$ and the $(N-2)$-bits information $c(k)$, the receiver uses (17)-(18) to compute the updated values of $\varepsilon_{min}(k)$ and $\varepsilon_{max}(k)$ and (19) to compute the encoded value $y_c(k)$ of $y_m(k)$. It then feeds the estimation algorithm by the new value $y_c(k)$.

3 Convergence results

In all the forthcoming developments, the following notation is used to denote the encoding error at instant $t_k = k\tau$

$$e(k) = y_m(k) - y_c(k) \tag{20}$$

while $\delta(k)$ is defined by

$$\delta(k) = y_m(k) - y_c(k-1) \tag{21}$$

note that the relevance of $\delta(k)$ comes from (15). From this, one can easily derive the following bounds on the encoding error

$$|e(k)| \leq \max\Big\{\delta(k) - \varepsilon_{max}(k-1)\ ,\ \frac{\varepsilon_{min}(k-1) + \varepsilon_{max}(k-1)}{2(2^{N-2}-1)}\Big\} \quad \forall\delta(k) \geq 0 \tag{22}$$

$$|e(k)| \leq \max\Big\{-\varepsilon_{min}(k-1) - \delta(k)\ ,\ \frac{\varepsilon_{min}(k-1) + \varepsilon_{max}(k-1)}{2(2^{N-2}-1)}\Big\} \quad \forall\delta(k) \leq 0 \tag{23}$$

The convergence results of any encoding scheme depends on the assumptions used to characterize the behaviour of the encoded signal. To this respect, two different characterizations are considered in the present paper

Characterization 1
The measurement $y_m(\cdot)$ is characterized by

$$y_m(k\tau) = y(k\tau) + v_1(k) \quad ; \quad \forall t \ , \ |\dot{y}(t)| \leq d_{11} \quad ; \quad v(k) \leq d_{12}(\tau) \tag{24}$$

where v_1 is the measurement noise that is assumed to have a p.d.f denoted by f_1^τ.

Characterization 2
The measurement $y_m(\cdot)$ is characterized by

$$y_m((k+1)\tau) = y_m(k\tau) + v_2(k) \quad ; \quad v_2(k) \text{ has a p.d.f } \quad f_2^\tau(\cdot) \text{ s.t } supp(v_2) = [-d_2(\tau), d_2(\tau)] \tag{25}$$

It is worth noting that the two above characterizations lead to a common property that may be expressed as follows

$$y_m(k+1) = y_m(k) + v(k) \quad ; \quad |v(k)| \leq d \tag{26}$$

provided that d is defined as follows

$$d := \begin{cases} d_{11}\tau + d_{12}(\tau) & \text{for characterization 1} \\ d_2(\tau) & \text{for characterization 2} \end{cases} \tag{27}$$

Throughout the present section the following choice is used for the parameter ε_0 appearing in (11)-(12)

$$\varepsilon_0 := \alpha^{-1}\beta d =: \alpha^{-1}d' \quad ; \quad \text{that is } d' := \beta d \quad ; \quad \beta > 1 \tag{28}$$

Consequently, the whole encoding scheme depends on the choice of the pair $(\alpha, \beta) \in]1, \infty[^2$.

Several lemmas are proved hereafter that are needed to establish the main results of this section.

Lemme 3.1 *For both characterizations, assume that the updating rules (11)-(12) are used with ε_0 given by (28). For all initial conditions*

$$\Big(\varepsilon_{min}(0), \varepsilon_{max}(0)\Big) \in \Big\{\alpha^j d' \quad | \quad j \in \mathbb{Z}\Big\}$$

the trajectory $\Big(\varepsilon_{min}(k), \varepsilon_{max}(k)\Big)_{k\in\mathbb{N}}$ reaches the nine - values invariant set

$$\mathcal{E}_d(\alpha, \beta) := \Big\{\alpha^{-1}d' \ , \ d' \ , \ \alpha d'\Big\} \times \Big\{\alpha^{-1}d' \ , \ d' \ , \ \alpha d'\Big\}$$

in finite time. ♣

Lemme 3.2 *For both characterizations, if the pair (α, β) satisfies the following inequality*

$$\beta > \max\Big\{\frac{2\alpha}{1+\alpha}, \frac{\alpha^2}{2\alpha - 1}\Big\} \tag{29}$$

then for all initial conditions $\delta(0) \in \mathbb{R}$ and all

$$\Big(\varepsilon_{min}(0), \varepsilon_{max}(0)\Big) \in \Big\{\alpha^j d' \quad | \quad j \in \mathbb{Z}\Big\}$$

there is a finite $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, the following holds

$$\delta(k) \in \Big[-\gamma d \ , \ \gamma d\Big] \tag{30}$$

where $\gamma > 0$ is given by

$$\gamma(\alpha, \beta) := \alpha\beta + (1 - \alpha^{-1}\beta) \tag{31}$$

♣

Based on lemmas 3.1-3.2, the following proposition that gives an asymptotic upper bound on the encoding error may be easily proved

Proposition 3.1 *For both characterizations, if the pair (α, β) satisfies the following inequality*

$$\beta > \max\Big\{\frac{2\alpha}{1+\alpha}, \frac{\alpha^2}{2\alpha - 1}\Big\} \tag{32}$$

then for all initial conditions $\delta(0) \in \mathbb{R}$ and all

$$\Big(\varepsilon_{min}(0), \varepsilon_{max}(0)\Big) \in \Big\{\alpha^j d' \quad | \quad j \in \mathbb{Z}\Big\}$$

there is a finite $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, the encoding error $e(k)$ satisfies the following inequality

$$\mid e(k) \mid \leq \max\Big\{\frac{\alpha\beta + (1 - \alpha^{-1}\beta)}{2^{N-2} - 1}\ ,\ \alpha\beta + (1 - 2\alpha^{-1}\beta)\Big\}d \tag{33}$$

Proposition 3.1 leads to the following corollary that characterizes the performance of the proposed scheme in a free-noise context

Corollary 3.1

1. *When using the above encoding scheme to encode a static variable y_0 with free-noise measurements, the encoding error converges to 0, namely*

$$\forall N > 2, \qquad \lim_{k\to\infty} e(k) = \lim_{k\to\infty} |y_0 - y_c(k)| = 0$$

2. *When using the above encoding scheme to encode a ramp $y_m(t) = rt$ with free-noise measurements, there is a finite k_0 such that for all $k \geq k_0$, one has*

$$\forall N > 2 \qquad \mid e(k) \mid \leq \max\Big\{\frac{\alpha\beta + (1 - \alpha^{-1}\beta)}{2^{N-2} - 1}\ ,\ \alpha\beta + (1 - 2\alpha^{-1}\beta)\Big\}r\tau$$

   in particular, $\forall N > 2$, the asymptotic encoding error tends to 0 with the sampling period τ.

All the results obtained so far hold for both characterizations since they are only based on the use of (26) in which d is defined by (27). The following result characterizes the stochastic behaviour of $(\varepsilon_{min}, \varepsilon_{max})$ for given p.d.f's f_i^τ characterizing the behaviour of v_i in (24) and (25). First, the following quantities need to be defined

$$p_1(\alpha, \beta) = \int_{-d'}^{-\alpha^{-1}d'} f^\tau(s)ds \tag{34}$$

$$p_2(\alpha, \beta) = \int_{-\alpha^{-1}d'}^{\alpha^{-1}d'} f^\tau(s)ds \tag{35}$$

$$p_3(\alpha, \beta) = \int_{\alpha^{-1}d'}^{d'} f^\tau(s)ds \tag{36}$$

The starting point is lemma 3.1 stating that after a finite $k_0 \in \mathbb{N}$, the trajectory $\Big(\varepsilon_{min}(k), \varepsilon_{max}(k)\Big)_{k\geq k_0}$ reaches the nine-values limit set $\mathcal{E}_d(\alpha, \beta)$. In order to easily refer to the elements of the terminal set $\mathcal{E}_d(\alpha, \beta)$, the following auxiliary variable is used

$$w := \begin{pmatrix} \log_\alpha(\frac{\varepsilon_{min}}{d'}) \\ \log_\alpha(\frac{\varepsilon_{max}}{d'}) \end{pmatrix} \in \mathbb{N}^2 \tag{37}$$

with the above notations, the set $\mathcal{E}_d(\alpha, \beta)$ is isomorphic to the set

$$\Big\{w \in \mathbb{Z}^2 \quad | \quad w \in \{-1, 0, 1\} \times \{-1, 0, 1\}\Big\}$$

hence denoted hereafter by the same notation, namely $\mathcal{E}_d(\alpha, \beta)$.

Now, using the definitions (34)-(36), it can be easily seen that the stochastic behaviour of $w(k)$ given by (37) can be described by the discrete-time Markov chain depicted on Figure 1 where the notation S_j for $j = 1, \ldots, 9$ has been used to number the 9 states of the final set $\mathcal{E}_d(\alpha, \beta)$. This amounts to define a bijection

$$S : \{1, \ldots, 9\} \to \mathcal{E}_d(\alpha, \beta)$$

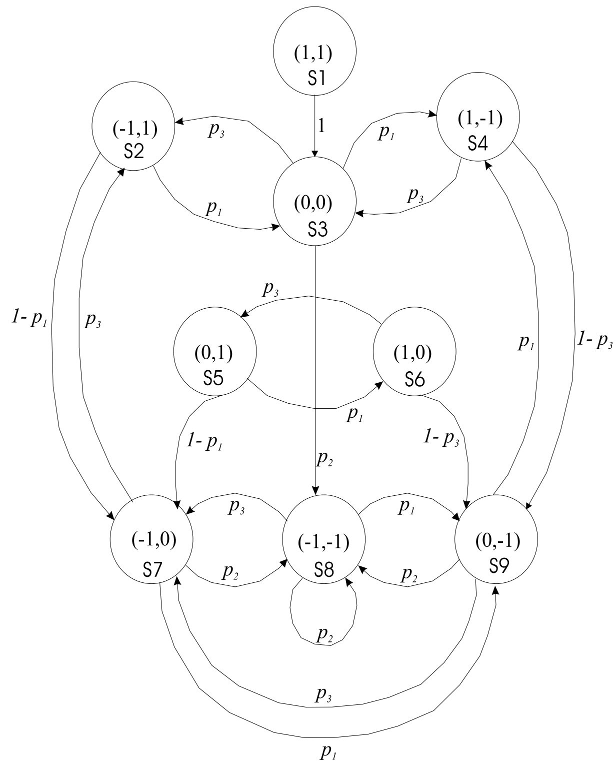

Figure 1: The Markov chain of the stochastic behaviour of w given by (37) in the final set $\mathcal{E}_d(\alpha, \beta)$. Note that the probability transitions p_j's denote $p_j(\alpha, \beta)$ given by (34)-(36)

and to use the notation $S_j = S(j)$. This numbering makes easier the use of Markov chains tools. Indeed, Markov chains are used to describe the dynamic behaviour of stochastic process taking place in discrete sets. Roughly speaking, a Markov chain is a graph in which each state denotes the process variable taking one of the admissible discrete values. Weighted arcs are then used to describe transitions between the states. The weight of the arcs are the probability that the corresponding transition takes place given the stochastic characterization of the dynamic process. In our case, the finite discrete set is precisely $\mathcal{E}_d(\alpha, \beta)$ and the corresponding Markov chain is depicted on Figure 1.

To show how the Markov chain of Figure 1 describes the behaviour of w (or equivalently $(\varepsilon_{min}(k), \varepsilon_{max}(k))$), let us take the transition between the state $S(3) = (0, 0)$ to $S(2) = (-1, 1)$ that is supposed to occur at some instant k. Therefore, one has by definition

$$\varepsilon_{min}(k-1) = \alpha^0 d^{'} = d^{'} \quad ; \quad \varepsilon_{max}(k-1) = \alpha^0 d^{'} = d^{'}$$

Furthermore,

$$\varepsilon_{min}(k) = \alpha^{-1} \varepsilon_{min}(k-1) \quad ; \quad \varepsilon_{max}(k) = \alpha \varepsilon_{min}(k-1) \tag{38}$$

Now, according to the updating rules (11)-(12), equation (38) holds if and only if

$$y_m(k) - y_m(k-1) = v(k) > \alpha^{-1} d^{'}$$

and the probability of this event is clearly $p_3(\alpha, \beta)$ defined by (36). The other transitions probabilities may be retrieved in a similar way.

Now, for a given Markov chain, the so-called transition matrix may be defined for computational purposes. In particular, the corresponding transition matrix for the Markov chain of figure 1 is clearly given by

$$M(\alpha,\beta) := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_1 & 0 & 0 & 0 & 1-p_1 & 0 & 0 \\ 0 & p_3 & 0 & p_1 & 0 & 0 & 0 & p_2 & 0 \\ 0 & 0 & p_3 & 0 & 0 & 0 & 0 & 0 & 1-p_3 \\ 0 & 0 & 0 & 0 & 0 & p_1 & 1-p_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & 1-p_3 \\ 0 & p_3 & 0 & 0 & 0 & 0 & 0 & p_2 & p_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & p_3 & \boxed{p_2} & p_1 \\ 0 & 0 & 0 & p_1 & 0 & 0 & p_3 & p_2 & 0 \end{pmatrix} \tag{39}$$

Recall that the transition matrix $M(\alpha,\beta)$ appears in the dynamic relation

$$\vartheta(k+1) = \vartheta(k)M(\alpha,\beta) \tag{40}$$

where

$$\vartheta(k) = \big(\vartheta_1(k) \quad \dots \quad \vartheta_9(k)\big) \in [0,1]^9 \quad ; \quad \vartheta_j(k) := \Pr\Big(w(k) = S_j\Big) \tag{41}$$

More precisely, (40) describes the dynamic evolution of the probability vector $\vartheta(k)$ whose j^{th} component is the probability that $\Big(\varepsilon_{min}(k), \varepsilon_{max}(k)\Big)$ is in the j^{th} state of the final set $\mathcal{E}_d(\alpha,\beta)$. (For more details about discrete-time Markov chain, see [8, 6]).

Now using classical Markov chains related tools and the particular structure of the Markov chain of Figure 1, the following result can be obtained

Proposition 3.2 *Under the updating rules (11)-(12), the probability vector $\vartheta(k)$ [defined by (37) and (41)] describing the stochastic behaviour of the encoding parameters ε_{min} and ε_{max} tends to a unique* **computable** *stationary value that is independent of the initial probability vector $\vartheta(0)$. More precisely*

$$\lim_{k\to\infty} \vartheta(k) = \vartheta^{\infty}(\alpha,\beta)$$

where $\vartheta^{\infty}(\alpha,\beta)$ is the unique solution of the linear system

$$\vartheta\Big[\mathbb{I} - M(\alpha,\beta)\Big] = 0 \quad ; \quad \vartheta \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \tag{42}$$

In particular the expectation of ε_{min} and ε_{max} are given by

$$\bar{\varepsilon}_{min} := \sum_{i=1}^{9} \vartheta_i^{\infty}(\alpha,\beta)\alpha^{S_1(i)} \quad ; \quad \bar{\varepsilon}_{max} := \sum_{i=1}^{9} \vartheta_i^{\infty}(\alpha,\beta)\alpha^{S_2(i)} \tag{43}$$

while the variances $\sigma^2_{\varepsilon_{min}}$ and $\sigma^2_{\varepsilon_{max}}$ are given by

$$\sigma^2_{\varepsilon_{min}} := \sum_{i=1}^{9} \vartheta_i^{\infty}(\alpha,\beta)\Big(\alpha^{S_1(i)} - \bar{\varepsilon}_{min}\Big)^2 \quad ; \quad \sigma^2_{\varepsilon_{min}} := \sum_{i=1}^{9} \vartheta_i^{\infty}(\alpha,\beta)\Big(\alpha^{S_2(i)} - \bar{\varepsilon}_{max}\Big)^2 \tag{44}$$

This result is used to optimize the choice of (α,β) as explained in the next section.

4 Choosing the encoding parameters (α,β)

Recall that when only the bounding interval $[-d,d]$ is used to characterize the difference $y_m(k+1) - y_m(k)$, the encoding error is proved to be such that (see proposition 3.2)

$$|e(k)| \le \max\Big\{\frac{\gamma(\alpha,\beta)}{2^{N-2}-1} \ , \ \gamma(\alpha,\beta) - \alpha^{-1}\beta\Big\} \tag{45}$$

where

$$\gamma(\alpha,\beta) := \alpha\beta + (1-\alpha^{-1}\beta)$$

Now the choice of (α,β) depends on the criterion to be minimized. Indeed, if one aims to minimize the worst case in terms of the encoding error, then the natural choice would be

$$(\hat{\alpha},\hat{\beta}) := \text{Arg}\min_{(\alpha,\beta)}\Big[\gamma(\alpha,\beta) \quad ; \quad \text{under} \quad \alpha = 1+\eta_1 \quad \& \quad \beta \geq \max\Big\{\frac{2\alpha}{1+\alpha},\frac{\alpha^2}{2\alpha-1}\Big\}+\eta_2\Big] \tag{46}$$

for some positive reals $\eta_1 \ll 1$ and $\eta_2 \ll 1$. A careful look on the expression clearly leads to the conclusion that in this case, the better choice is clearly such that one obtains a strictly decreasing sequence $\beta \approx \alpha \approx 1$. Consequently, the expectation of ε_{min} and ε_{max} are such that

$$\bar{\varepsilon}_{min}(\hat{\alpha},\hat{\beta}) \approx \bar{\varepsilon}_{max}(\hat{\alpha},\hat{\beta}) \approx d \tag{47}$$

This choice may be suitable when the underlying observer-based feedback control is designed us-

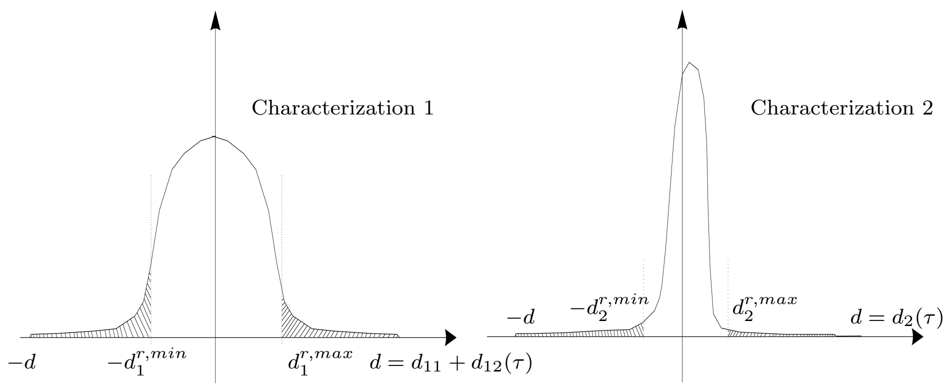

Figure 2: Forms of the p.d.f of $y_m(k+1) - y_m(k)$ for which $(\hat{\alpha},\hat{\beta})$ given by (46) may be very pessimistic.

ing robustness min-max related tools. However, it may be quite pessimistic when the p.d.f f_i^τ for characterization i is such that there are $d_{i,min}^r$ and $d_{i,max}^r$ **much smaller than d** such that

$$\int_{-d_i^{r,min}(\tau)}^{d_i^{r,max}(\tau)} f_i^\tau(\sigma)d\sigma > 0.99 \tag{48}$$

the corresponding situation is depicted on figure 2. In this case, one would like to have (47) satisfied with $d_i^r(\tau)$ replacing d. This can be expressed by using the following choice for (α,β)

$$(\hat{\alpha},\hat{\beta}) := \text{Arg}\min_{(\alpha,\beta)}\Big[\Big(\bar{\varepsilon}_{min}(\alpha,\beta) - d_i^{r,min}(\tau)\Big)^2 + \Big(\bar{\varepsilon}_{max}(\alpha,\beta) - d_i^{r,max}(\tau)\Big)^2\Big] \tag{49}$$

under the constraints

$$\begin{aligned}
&\big(\bar{\varepsilon}_{min}(\alpha,\beta),\bar{\varepsilon}_{min}(\alpha,\beta)\big) \quad \text{given by (42), (43)}\\
&\alpha \geq 1+\eta_1 \quad \& \quad \beta \geq \max\Big\{\frac{2\alpha}{1+\alpha},\frac{\alpha^2}{2\alpha-1}\Big\}+\eta_2\\
&\bar{\varepsilon}_{max}(\alpha,\beta) \geq d_i^{r,max}(\tau)\\
&\bar{\varepsilon}_{min}(\alpha,\beta) \geq d_i^{r,min}(\tau)
\end{aligned}$$

from some positive small reals $\eta_1 \approx 0$ and $\eta_2 \approx 0$. Indeed, this simply use the min-max strategy but on the somehow "*effective*" values of d.

5 Illustrative examples

In this section, some illustrative simulations are proposed to show the effectiveness of the proposed encoding scheme. With this respect, the p.d.f depicted in figure 3 is used to generate the signals $v_1(\cdot)$ or $v_2(\cdot)$ appearing in (24) [resp. (25)] according to the characterization being considered.

$f_1^\tau = f_2^\tau$

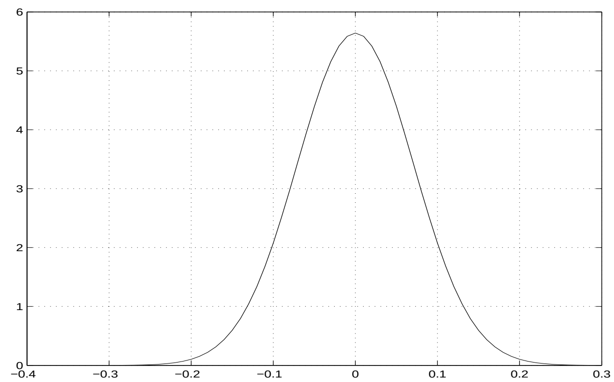

Figure 3: The p.d.f used in the generation of the signals $v_1(k\tau)$ and $v_2(k\tau)$ appearing in the measured signal characterizations (24) and (25)

Note that the used p.d.f is such that the values of d_{12} and d_2 in (24) and (25) respectively are given by

$$d_{12}(\tau) = d_2(\tau) = 0.3$$

Simulations are done with $\tau = 0.05$ and using the following scenarios

1. two scenarios with characterization 1
   (a) $y_m(k\tau) = 5\sin(k\tau) + v(k)$. In this case, according to (27), one has
   $$d = |\dot{y}|_{max}\tau + d_{12}(\tau) = 5 \times 0.05 + 0.3 = 0.55$$
   (b) $y_m(k\tau) = 0.2k\tau + v(k)$. In this case, according to (27), one has
   $$d = |\dot{y}|_{max}\tau + d_{12}(\tau) = 0.2 \times 0.05 + 0.3 = 0.31$$
2. A scenario using the second characterization, namely $y_m(k+1) = y_m(k) + v(k)$. In this case, one clearly has $d = d_2(\tau) = 0.3$. Furthermore, it can be shown by simple computations that the value
   $$d^{r,min} = d^{r,max} \approx 0.182 \tag{50}$$
   satisfies the inequality (48). This is used in the remainder of this section in the validation of the heuristic optimization-based approach proposed in section 4.

Figures 4 and 5 shows the performance of the proposed encoding scheme when characterization 1 is used. In each case, the top plot shows the measured and the decoded value y_c. The middle plot shows the evolution of the encoding error $e = y_m - y_c$ in the beginning of the simulation. This enables the transient response to be clearly observed. The theoretical error bounds stated in proposition 3.1 are also plotted in order to show that the encoding error effectively reaches the corresponding bounding set after a finite number of iterations. The parameter values used in the above simulations are given by

$$\alpha = 1.5 \quad ; \quad N = 3$$

namely, two bits are used to communicate the updating information $(i_{min}, i_{max}) \in \mathcal{I}$ while one bit is used to encode the measurement as explained in the preceding sections.

In order to use the stochastic characterization of the noise given by the p.d.f depicted in figure 3, the result of proposition 3.2 has been used to compute the expectations $(\bar{\varepsilon}_{min}(\alpha,\beta), \bar{\varepsilon}_{min}(\alpha,\beta))$ as well as the variances $(\sigma^2_{\varepsilon_{min}}(\alpha,\beta), \sigma^2_{\varepsilon_{min}}(\alpha,\beta))$ for different values of $\alpha \in]1, 20]$ and for β given by

$$\beta = \max\left\{\frac{2\alpha}{1+\alpha}, \frac{\alpha^2}{2\alpha - 1}\right\} \tag{51}$$

The results are given on figure 6 where it can be also observed that the solution of (49) is approximatively given by

$$\hat{\alpha} \approx 3 \quad ; \quad \hat{\beta} \approx 1.5$$

With this choice, the error bound given by proposition 3.1 can be estimated to be approximately equal to 1.5.

In order to appreciate the quality of the heuristic solution suggested in section 4, the two following simulations are proposed

- Figure 7: Simulation for the worst case approximate optimal choice $\alpha = 1.05$.
- Figure 8: Simulation of the heuristically optimal choice $\hat{\alpha} \approx 3$ proposed in section 4.

The comparison of these two figures suggests the following remark

- As expected The **theoretical** bounds for the encoding error are clearly greater for the higher value $\hat{\alpha} = 3$ of the encoding parameter α.
- However, the effective time evolution of the resulting encoding error is far from these bounds. Indeed, these bounds are based on the potentially possible high values of v that are very unlikely to occur.
- To attest the intuition that underlies the heuristic rule proposed in section 4, a detailed plot of the **asymptotic** phase (namely, transient phase has been discarded) of the encoding error during these two simulations are given in Figure 9 where it can be clearly observed that the heuristically optimized choice $\hat{\alpha} = 3$ leads to mean encoding error that is lower than the one corresponding to the min-max approximate choice $\alpha = 1.05$. The mean values of the absolute values of the encoding error in these two cases (after discarding the transient) are given by

$$\frac{1}{k_f - k_0 + 1}\Big[\sum_{k=k_0}^{k=k_f} |e(k)|\Big]_{\alpha=1.05} \approx 0.15 \quad ; \quad \frac{1}{k_f - k_0 + 1}\Big[\sum_{k=k_0}^{k=k_f} |e(k)|\Big]_{\alpha=\hat{\alpha}=3} = 0.09$$

  where k_0 and k_f delimit the post transient phases.

6 Conclusion

In this paper, a new equal-partition scheme is proposed for measurement encoding in NCS's. The proposed scheme is characterized by low-cost computation to be performed by the sensor device. Furthermore, the encoding scheme is defined via a Markovian process enabling stochastic properties of the noise signal to be explicitly handled. Convergence results have been given to characterize the asymptotic behaviour in terms of upper bounds on the encoding error. Furthermore, it has been shown that the resulting Markovian process admits a unique and computable stationary solution that is used to characterize the stochastic behaviour of the bounding interval leading to a somehow heuristic optimization based procedure to choose the encoding parameters. The results have been illustrated through simulation.

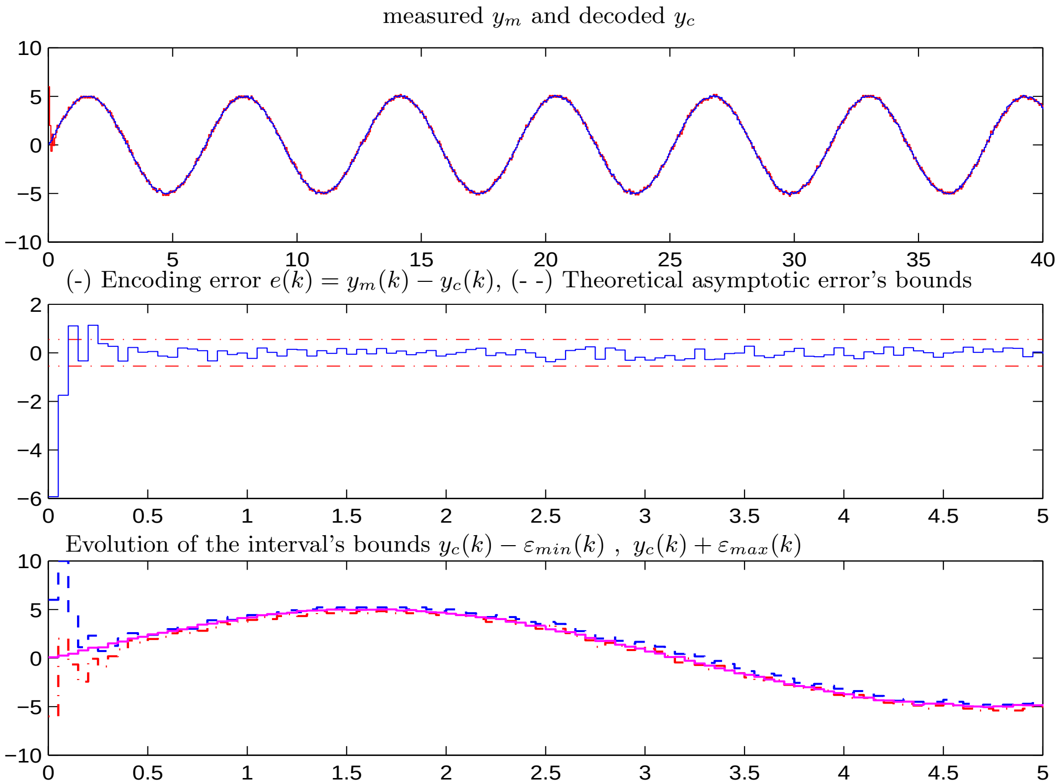

Figure 4: Case of sinusoidal noisy measured signal. $y_m(k\tau) = 5\sin(k\tau) + v(k)$. [$\alpha = 1.5$, $\tau = 0.05$ and $\varepsilon_{min}(0) = \varepsilon_{max}(0) = 6$ while $N = 3$]

A Appendix

A.1 Proof of lemma 3.1

Straightforward. Indeed, The attractiveness results from the following implications

$$\begin{aligned}
\Big\{\varepsilon_{max}(k-1) > \alpha d'\Big\} \quad &\Rightarrow \quad \Big\{\varepsilon_{max}(k) = \alpha^{-1}\varepsilon_{max}(k-1)\Big\} \\
\Big\{\varepsilon_{min}(k-1) < -\alpha d'\Big\} \quad &\Rightarrow \quad \Big\{\varepsilon_{min}(k) = \alpha^{-1}\varepsilon_{min}(k-1)\Big\}
\end{aligned}$$

while the invariance is a direct consequence of

$$\begin{aligned}
\Big\{\varepsilon_{max}(k-1) = \alpha d'\Big\} &\Rightarrow \Big\{\varepsilon_{max}(k) = d'\Big\} \\
\Big\{\varepsilon_{min}(k-1) = -\alpha d'\Big\} &\Rightarrow \Big\{\varepsilon_{min}(k) = d'\Big\}
\end{aligned}$$

$\diamondsuit$

A.2 Proof of lemma 3.2

The proof is rather classical and lies in the two following facts

1. There is a finite $k^* \in \mathbb{N}$ such that $\delta(k^*)$ meets the set $[-\alpha d' \ , \ \alpha d']$
2. Any trajectory of δ that starts in $[-\alpha d' \ , \ \alpha d']$ remains in $[-\gamma d \ , \ \gamma d]$

Proof of 1. Assume that $\delta(i) > \alpha d'$ for all $i \in \mathbb{N}$. We shall prove that this leads to a contradiction. Take k sufficiently high for the final set $\mathcal{E}(\alpha, \beta)$ to be definitely reached by the trajectory $\Big(\varepsilon_{min}(i), \varepsilon_{max}(i)\Big)_{i \geq k}$ [see lemma 3.1].

Assume that $\delta(k) > \alpha d'$, one has by definition

$$\begin{aligned}
\delta(k+1) &= y_m(k+1) - y_c(k) = v(k) + y_m(k) - y_c(k) \\
&= \delta(k) + v(k) - [y_c(k) - y_c(k-1)] \\
&\leq \delta(k) + v(k) - \varepsilon_{max}(k-1) \quad \text{because } y_m(k) > \alpha d' \geq \epsilon_{max}(k-1) \qquad (52)
\end{aligned}$$

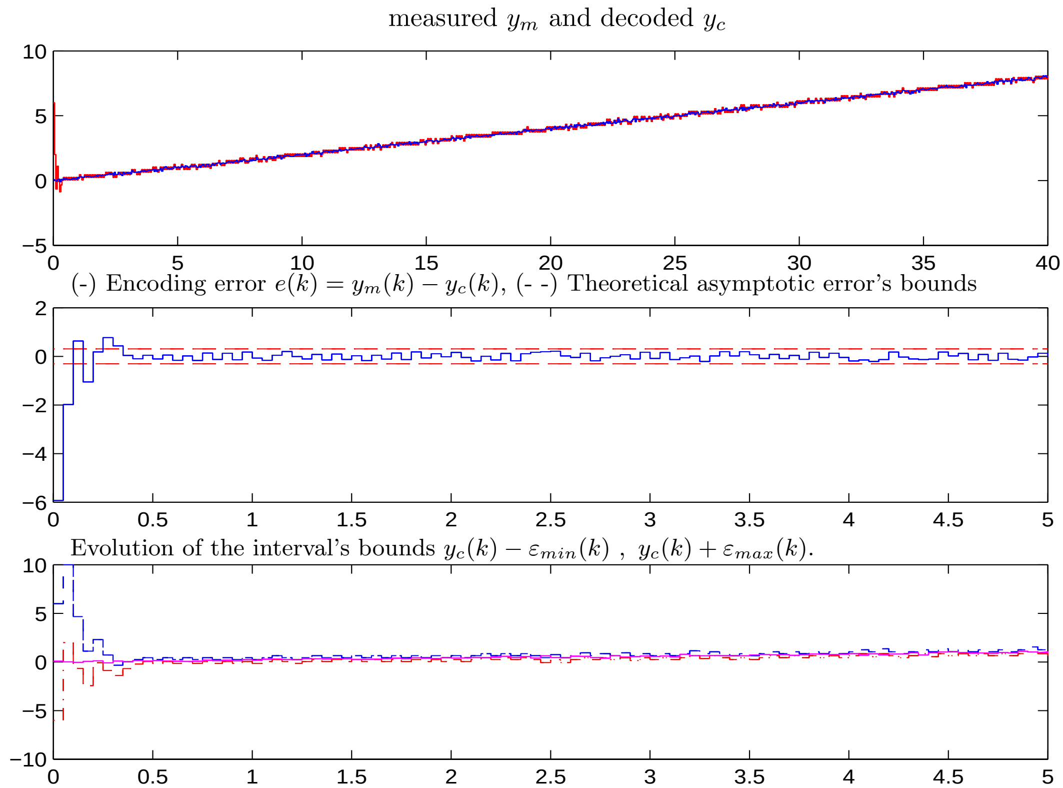

Figure 5: Case of ramp noisy measured signal. $y_m(k\tau) = 0.2k\tau + v(k)$. [$\alpha = 1.5$, $\tau = 0.05$ and $\varepsilon_{min}(0) = \varepsilon_{max}(0) = 6$ while $N = 3$]

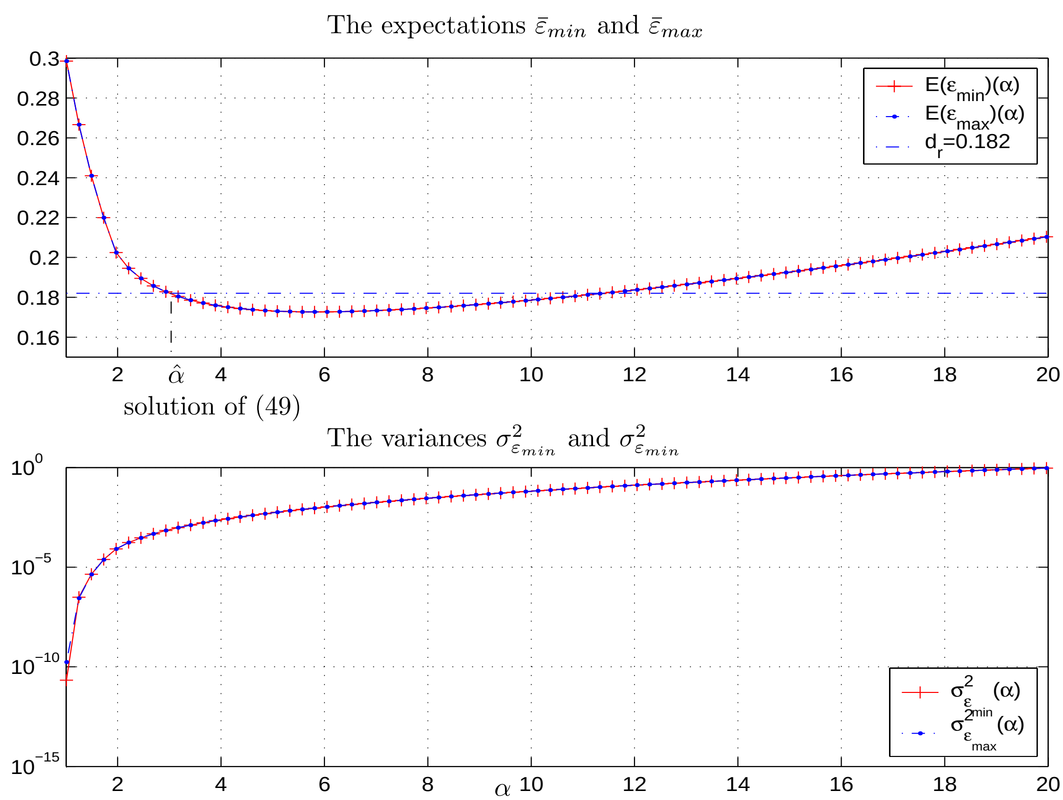

Figure 6: Evolution of the expectations $(\bar{\varepsilon}_{min}(\alpha, \beta), \bar{\varepsilon}_{min}(\alpha, \beta))$ and the variances $(\sigma^2_{\varepsilon_{min}}(\alpha, \beta), \sigma^2_{\varepsilon_{min}}(\alpha, \beta))$ when the encoding parameter α varies in $]1, 20]$ while β is given by equation (51)

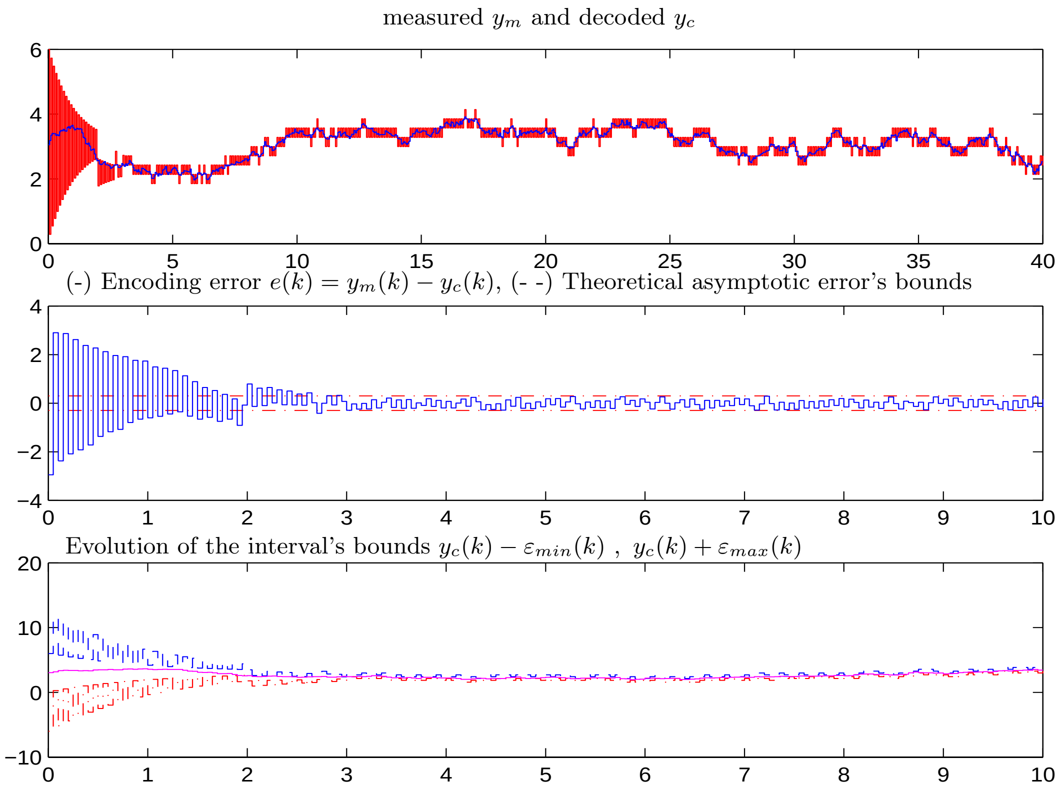

Figure 7: Case of characterization 2 $y_m(k+1) = y_m(k) + v(k)$ with worst-case based optimal choice $\alpha = 1.05$]

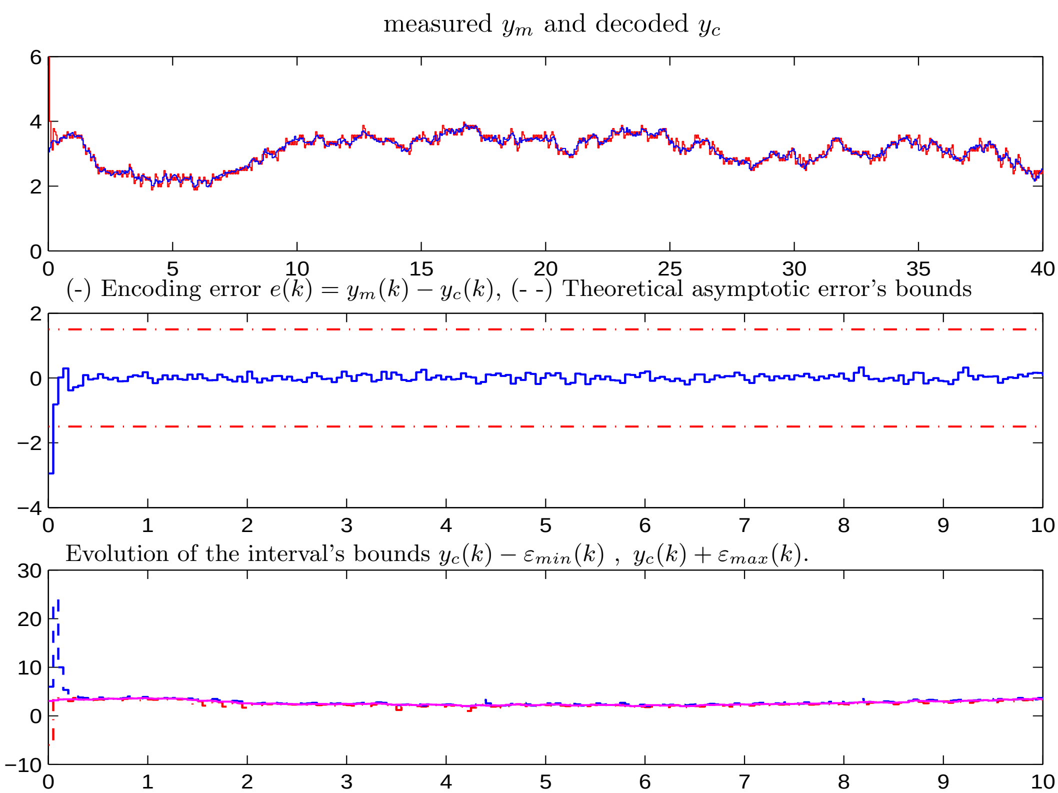

Figure 8: Case of characterization 2 $y_m(k+1) = y_m(k) + v(k)$ with the proposed heuristic optimal choice $\hat{\alpha} = 3$]

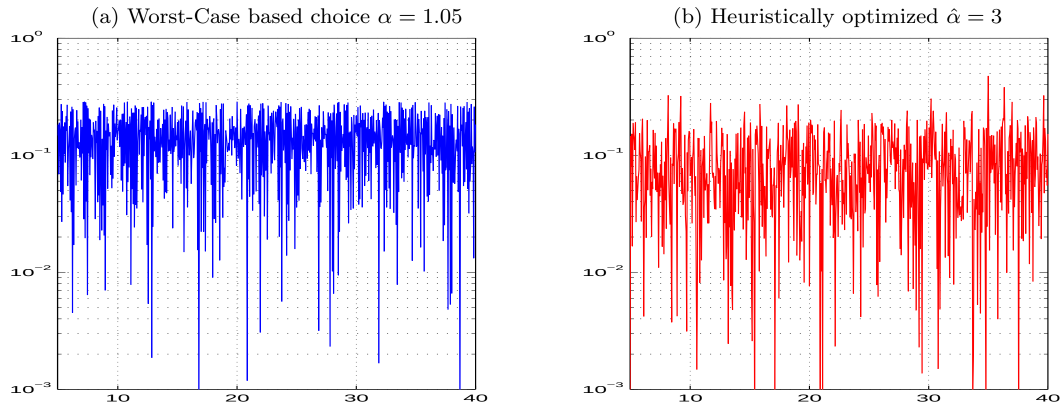

Figure 9: Comparison between post-transient behaviours under (a) the worst-case-based choice $\alpha = 1.05$ and (b) the heuristically optimized choice $\hat{\alpha} = 3$]

two cases may be distinguish

- If $\epsilon_{max}(k-1) \in \{d', \alpha d'\}$, then since $v(k) \le d$, inequality (52) implies that

$$\delta(k+1) \le \delta(k) - (\beta - 1)d \tag{53}$$

- If $\epsilon_{max}(k-1) = \alpha^{-1}d'$ then by writing (52) at $k+1$, one obtains

$$\begin{aligned} \delta(k+2) &\le \delta(k+1) + v(k+1) - \epsilon_{max}(k) \\ &\le \delta(k) + v(k) + v(k+1) - \varepsilon_{max}(k-1) - \varepsilon_{max}(k) \end{aligned} \tag{54}$$

Now according to the value of $v(k)$, two situations may be investigated

  - **Either** $v(k) \le \alpha^{-2}d'$ in which case, one has $\epsilon_{max}(k) = \alpha^{-1}d'$ and (54) gives

$$\begin{aligned} \delta(k+2) &\le \delta(k) + \alpha^{-2}d + d - 2\alpha^{-1}d' \\ &\le \delta(k) + \Big[1 - \alpha^{-2}(2\alpha - 1)\beta\Big]d \end{aligned}$$

but according to (29), there is some $\sigma_1 > 0$ such that

$$1 - \alpha^{-2}(2\alpha - 1)\beta = -\sigma_1 d$$

therefore,

$$\delta(k+2) \le \delta(k) - \sigma_1 d \tag{55}$$

  - **Or** $v(k) > \alpha^{-2}d'$ in which case one has $\varepsilon_{max}(k) = d'$ and (54) gives

$$\begin{aligned} \delta(k+2) &\le \delta(k) + 2d - [\alpha^{-1} + 1]\beta d \\ &\le \delta(k) + \Big[2 - \frac{1+\alpha}{\alpha}\beta\Big]d \end{aligned} \tag{56}$$

gain, according to (29) there must be some $\sigma_2 > 0$ such that

$$\Big[2 - \frac{1+\alpha}{\alpha}\beta\Big] = -\sigma_2$$

Therefore

$$\delta(k+2) \le \delta(k) - \sigma_2 d \tag{57}$$

From equations (53), (55) and (57) it can be inferred that there is a strictly increasing subsequence $(k_j)_{j\in\mathbb{N}}$ such that

$$\delta(k_{j+1}) \le \delta(k_j) - \min\{\beta - 1, \sigma_1, \sigma_2\}d$$

which clearly contradicts the assumption $\delta(i) > \alpha d'$ for all $i \in \mathbb{N}$. Using exactly the same development, one can show that the $\delta(i)$ cannot remain continuously $< -\alpha d'$. This ends the proof of fact1, namely, there is a finite k^* such that $\delta(k^*) \in [-\alpha d', \alpha d']$.

proof of fact2. Assume that $\delta(k) \in [-\alpha d', \alpha d']$. The arguments used in the proof of fact1 clearly show that an excursion outside the interval $[-\alpha d', \alpha d']$ is possible only in the two following cases

- **Either** $\varepsilon_{max}(k-1) = \alpha^{-1}d'$. In this case the maximal excursion is given by

$$\delta(k+1) = \delta(k) + v(k) - [y_c(k) - y_c(k-1)]$$

in which the worst configuration $v(k) = d$ and $\delta(k) = \alpha d'$ is used, namely

$$\delta(k+1) = \alpha d' + d - \alpha^{-1}d' = \Big[\alpha\beta + (1-\alpha^{-1})\beta\Big]d = \gamma d$$

- **or** $\varepsilon_{min}(k-1) = \alpha^{-1}d'$. In this case the maximal excursion is given by

$$\delta(k+1) = \delta(k) + v(k) - [y_c(k) - y_c(k-1)]$$

in which the worst configuration $v(k) = -d$ and $\delta(k) = -\alpha d'$ is used, namely

$$\delta(k+1) = -\alpha d' - d + \alpha^{-1}d' = -\Big[\alpha\beta + (1-\alpha^{-1})\beta\Big]d = -\gamma d$$

This clearly ends the proof of lemma 3.2 $\diamondsuit$

A.3 Proof of proposition 3.1

Take k sufficiently high for the consequences of lemma 3.1 to hold. Let us discuss the case $0 \le \delta(k) \le \gamma d$. The opposite case follows from identical arguments. The encoding error has two expressions depending on whether $\delta(k) \le \varepsilon_{max}(k-1)$ or not.

- If $\delta(k) \le \varepsilon_{max}(k-1)$, $|e(k)|$ is given by

$$\mid e(k) \mid = \frac{\varepsilon_{min}(k-1) + \varepsilon_{max}(k-1)}{2(2^{N-2}-1)} \le \frac{2\gamma d}{2(2^{N-2}-1)} \tag{58}$$

- If $\delta(k) > \varepsilon_{max}(k-1)$, $|e(k)|$ is given by

$$\mid e(k) \mid = \delta(k) - \varepsilon_{max}(k-1) \le (\gamma - \alpha^{-1}\beta)d \tag{59}$$

Equations (58)-(58) clearly give the result using the expression (31) of γ. $\diamondsuit$

A.4 Poof of proposition 3.2

The fact that there exists an asymptotic solution to the dynamic equation (40) comes from the fact that $M(\alpha,\beta)$ admits one nonzero diagonal element (only one state has a reflexive arc, namely S_8, see figure 1). The uniqueness of the stationary state comes from the fact that the Markov chain under interest has only one ergodic component [8, 6] which is the trivial whole Markov chain itself except the first state S_1. Finally (42) simply states that the unique stationary solution of (40) is necessarily a left eigenvector of $\mathbb{I} - M(\alpha,\beta)$. Furthermore, being a probability distribution, it necessarily satisfies $\sum_{j=1}^{9} \vartheta_j = 1$. $\diamondsuit$

References

[1] Special issue on networks and control. IEEE CONTROL SYSTEM MAGAZINE., February 2001.

[2] Mazen Alamir, *Optimization based non-linear observers revisited.*, Int. J. Control **72** (1999), no. 13, 1204–1217.

[3] Dimitri P. Bertsekas, *Incremental least squares methods and the extended Kalman filter.*, SIAM J. Optim. **6** (1996), no. 3, 807–822.

[4] E. Busvelle and J.P. Gauthier, *High-gain and non-high-gain observers for nonlinear systems.*, anzaldo-meneses, a. (ed.) et al., contemporary trends in nonlinear geometric control theory and its applications. ed., pp. 257–286, Singapore: World Scientific, Singapore, 2002.

[5] D. F. Delchamps, *Stabilizing a linear system with quantized state feedback*, IEEE Transactions on Automatic Control **35** (1990), 916–924.

[6] Hubert Hennion and Loic Hervé, *Limit theorems for markov chains and stochastic properties of dynamical systems by quasi-compactness.*, Lecture Notes in Mathematics., Springer, Berlin, 2001.

[7] R. Jain, T. Şimşek, and P. Varaiya, *Control under communication constraints*, Proceedings of the IEEE Conference on Decision and Control (Las Vegas), 2002.

[8] J. G. Kemeny and J. L. Snell, *Finite markov chains*, Undergraduated texts in mathematics, Springer Verlag, 1976.

[9] Hassan K. Khalil, *High-gain observers in nonlinear feedback control.*, lect. notes control inf. sci. ed., pp. 249–268, Springer, London, 2002.

[10] X. Li and W. S. Wong, *State estimation with communication constraint*, Systems & Conrol Letters **28** (1996), 49–54.

[11] Hannah Michalska and David Q. Mayne, *Moving horizon observers and observer-based control.*, IEEE Trans. Autom. Control **40** (1995), no. 6, 995–1006.

[12] H. C. Papadopoulos, G. W. Wornell, and A.V. Oppenheim, *Sequential signal encoding from noisy measurements using quantizers with dynamic bias control*, IEEE Transactions on Information Theory **47** (2001), no. 3, 978–1002.

[13] Jean Picard, *Efficiency of the extended Kalman filter for nonlinear systems with small noise.*, SIAM J. Appl. Math. **51** (1991), no. 3, 843–885.

[14] Christopher V. Rao and James B. Rawlings, *Nonlinear moving horizon state estimation.*, prog. syst. control theory. ed., pp. 45–69, Birkhuser., Basel, 2000.

[15] T. Ushio and C. S. Hsu, *Chaotic rounding error in digital control systems*, IEEE Trans. Circuits and Systems **34** (1987), 133–139.

[16] G. C. Walsh, O. Beldiman, and L. G. Bushnell, *Error encoding algorithms for networked control systems*, Automatica **38** (2002), 261–267.

[17] G. C. Walsh and H. Ye, *Scheduling of network control systems*, IEEE Control System Magazine **21** (2001), no. 1, 57–65.

[18] Z. Wei, M. S. Branicky, and S. M. Phillips, *Stability of networked control systems*, IEEE Control System Magazine **21** (2001), no. 1, 84–99.

[19] D. Williamson, *Finite wordlength design of digital kalman filters for state estimation*, IEEE Transactions on Automatic Control **30** (1990), 930–939.

[20] W. S. Wong and R. W. Brockett, *Systems with finite communication bandwidth constraints – i: State estimation problems*, IEEE Transactions on Automatic Control **42** (1997), 1294–1298.

[21] ———, *Systems with finite communication bandwidth constraints – ii: Stabilization with limited information feedback*, IEEE Transactions on Automatic Control **44** (1999), no. 5, 1049–1053.