AN EFFICIENT ALGORITHM TO SOLVE OPTIMAL CONTROL PROBLEMS FOR NONLINEAR SWITCHED HYBRID SYSTEMS

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Abstract: This paper focuses on the gain in efficiency that may be obtained when using strong variations-like algorithms to solve optimal control problems for switched nonlinear systems. A simple version of a strong variations algorithm is proposed together with some new convergence results. An illustrative example is used to illustrate the efficiency of the proposed algorithm.

Keywords: Optimal control, switched hybrid system, maximum principle, convergence results.

1. INTRODUCTION

Hybrid systems paradigm is a key issue in process engineering. Many works have been done to properly define such systems (Branicky *et al.*, 1998; Piccoli, 1998). Then attention has been focused on possible characterizations of optimality enabling an exhaustive search over the whole set of possibilities to be avoided. Such characterizations may be directly obtained by using the Bellman principle via a dynamic programming approach (Bather, 2000; Hubert *et al.*, 2001; Hedlund and Rantzer, 2002; Branicky and Hebbar, 1999). Although such characterization is universal and hence directly applies to hybrid systems, its use for systems with high state's dimension is cumbersome since the complexity of the unknown function V increases exponentially with the state dimension in the general nonlinear case. That is the reason why generalizations of the Pontryagin Maximum Principle (**MP**) have been attempted. (Sussmann, 1999; Piccoli, 1998).

Another approach that gains increasing number of followers everyday is the one based on Mixed-Integer Logical Dynamics (**MLD**) formulation where basically linear dynamics are used with extended state containing the continuous and binary variables (Bemporad *et al.*, 2000).

In ((Xu and Antsaklis, 2002; Bemporad *et al.*, 2002; Shaikh and Caines, 2003; Shen and Caines, 2002), a two stage approach has been proposed. In the first stage, the total number of switches is a priori fixed as well as the sequence of active subsystems. By doing this, the cost function is only function of the switching instants. Therefore, finding the optimal switching instants for the given number of switches and the sequence of active subsystems is a classical nonlinear programming problem (Pierre, 1986). In the second stage, the a priori given data, namely, the number of switches and the sequence of active subsystems are updated to improve the optimal solution that would be obtained by the first stage. For a recent and almost exhaustive recent survey of existing algorithms, see (Xu and Antsaklis, 2003).

In this paper, it is shown that as long as switched nonlinear systems with only external switching controls are concerned, strong variations algorithms enables a unified approach that iterates on both continuous and logical variables and avoid the use of a priori assumptions on the number of switches or the sequence of active configurations. It is worth noting however that, like all iterative schemes that are inspired by the first order necessary conditions associated to the maximum principle, only extremal trajectories are looked for without any distinction between local or global minimizers. However, it is widely known form applied mathematicians experiences that, even so, such methods generally lead to good solutions to many practical problems.

The paper is organized as follows: First the problem in question is clearly stated 2. In section 3, a simple algorithm is proposed with some convergence results. Finally, in section 4, the algorithm is applied on an illustrative example of switched nonlinear systems to assess its efficiency in solving the optimal control problem under concern.

2. PROBLEM STATEMENT

Consider a switched nonlinear system given by

$$\dot{x} = f_q(x, v, t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $v \in \mathcal{V} \subset \mathbb{R}^m$ is the control input that belongs to some compact admissible subset $\mathcal{V}$. $q \in \underline{Q} := \{1, \ldots, Q\}$ is the configuration index. The present paper is concerned with the following optimal control problem

$$\min_{q(\cdot), v(\cdot)} \quad J(q(\cdot), v(\cdot)) := \int_{t_0}^{t_f} L_{q(\tau)}\Big(x(\tau), v(\tau), \tau\Big) d\tau$$

Let $\mathcal{A} \subset \{0,1\}^Q$ be the set of all $\alpha := (\alpha_1, \ldots, \alpha_Q) \in \{0,1\}^Q$ such that $\sum_{q=1}^{Q} \alpha_q = 1$. By concatenating (v, α) in a single vector $u := (v, \alpha) \in U := \mathcal{V} \times \mathcal{A}$, it clearly comes that the class of switched systems defined above may be put in the following general form

$$\dot{x} = f(x, u, t) := \sum_{q \in \underline{Q}} \alpha_q f_q(x, v, t) \tag{2}$$

in which the control input belongs to the non convex set U. Similarly, the cost function becomes (with $L(x, u, t) := \sum_{q=1}^{Q} \alpha_q L_q(x, v, t)$) :

$$J(u) := \int_{t_0}^{t_f} L(x(\tau), u(\tau), \tau) d\tau \tag{3}$$

3. THE PROPOSED ALGORITHM

In this section, a general algorithm is proposed to solve the optimal control problem (2)-(3) over the non convex set U of admissible controls. For, let us first give the working assumptions

Assumption 1. The functions f and L are twice continuously differentiable w.r.t x. Furthermore, there exists some $M > 0$ such that for all admissible control u, the corresponding state trajectory $x(t; t_0; x_0; u)$ satisfies $\|x(t; t_0; x_0; u)\| \leq M$ for all $t \in [t_0, t_f]$ and all initial states of interest x_0. Finally, the optimal solutions of (2)-(3) are normal (see (Pierre, 1986)).

Recall that under the above assumption, optimal profiles $(u^*(\cdot), x^*(\cdot))$ necessary satisfy a.e. on $[t_0, t_f]$ and for some non vanishing adjoint state profile $\lambda^*(\cdot)$ (with $\lambda^*(t_f) = 0$) :

$$\dot{x}^*(t) = f(x^*(t), u^*(t), t) \tag{4}$$

$$\dot{\lambda}^* = -H_x(x^*(t), u^*(t), \lambda^*(t), t) \tag{5}$$

$$H(x^*(t), u^*(t), \lambda^*(t), t) = \min_{u \in U} H(x^*(t), u, \lambda^*(t), t) \tag{6}$$

where $H(x, u, \lambda, t) := L(x, u) + \lambda^T f(x, u)$ is the problem's Hamiltonian.

Let $h = (t_f - t_0)/(N-1) > 0$ be a "*small*" sampling period and define the sampling instants by $t_{k+1} = t_k + h$. Any $\bar{u} := (\bar{u}(1), \ldots, \bar{u}(N-1)) \in \bar{U} := U \times \cdots \times U$ is hereafter identified to the corresponding piece-wise constant control profile defined over $[t_0, t_f]$ by $u(t_k + \tau) = \bar{u}(k)$ for all $k \in \{1, \ldots, N\}$ and all $\tau \in [0, h[$. Similarly, Given some $\bar{u}$, the finite dimensional approximations $\bar{x}, \bar{\lambda} \in \mathbb{R}^{Nn}$ are defined by integrating (4) using a second order Runge-Kutta method, this is shortly written as follows (with $\bar{x}(1) = x_0$ and $\bar{\lambda}(N) = 0$)

$$\bar{x}(k+1) = RK_2^F(\bar{x}(k), \bar{u}(k), k) \tag{7}$$

$$\bar{\lambda}(k-1) = RK_2^B(\bar{\lambda}(k), \bar{x}(k), \bar{u}(k), \cdots \bar{x}(k-1), \bar{u}(k-1), k) \tag{8}$$

where RK_2^F and RK_2^B are respectively a forward and a backward integration schemes. Finally, the following approximation of the performance index is used

$$\bar{J}(\bar{u}) = h \sum_{k=1}^{N-1} L\big(\bar{x}(k), \bar{u}(k), t_k\big) \tag{9}$$

Given the above definition, consider the following algorithm

- **Step 0**: Fix some small $\epsilon_u > 0, \epsilon_J > 0$, some integer i_{max} and two reals $d\mu > 0$ and $\gamma > 1$. Choose $\mu^0 \geq 0$ and some initial admissible guess $\bar{u}^0 \in \bar{U}$.
- **Step 1**: Compute $\bar{x}^0$ solution of (7) with $\bar{u} = \bar{u}^0$, let $i = 1$,
- **Step 2**: Compute $\bar{\lambda}^{i-1}$ solution of (8) with $\bar{u}^{i-1}$ and $\bar{x}^{i-1}$
- **Step 3**: Compute $\bar{u}^i$ and $\bar{x}^i$ such that
  - $\cdot$ $\bar{x}^i$ is solution of (7) with $\bar{u} = \bar{u}^i$ such that

- $\bar{u}^i(k) := \text{Arg} \min_{u \in U} \Big[H\big(\bar{x}^i(k), u, \bar{\lambda}^{i-1}(k), t_k\big) + \mu^{i-1} \parallel u - \bar{u}^{i-1}(k) \parallel^2 \Big]$,

- **Step 4**: **If** $\Big(\bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J\Big)$ and $\Big(\parallel \bar{u}^i - \bar{u}^{i-1} \parallel > \epsilon_u\Big)$ **then** let $\mu^{i-1} = max\Big(\mu^{i-1} + d\mu, \gamma\mu^{i-1}\Big)$ and return to **Step 3**,
- **Step 5**: **If** $\Big(\bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J\Big)$ then $\mu^i = \max\Big(0, \min\big(\mu^{i-1} - d\mu, \mu^{i-1}/\gamma\big)\Big)$,
- **Step 6**: **If** $\Big(\parallel \bar{u}^i - \bar{u}^{i-1} \parallel \leq \epsilon_u\Big)$ and $\Big(i \geq i_{max}\Big)$ **Then** stop **else** let $i = i+1$ and return to **Step 2**.

3.1 Convergence results

Proposition 1. there are positive reals $r > 0$ and $\sigma > 0$ such that, for sufficiently small step size h, the solutions of successive iterations satisfy the following inequality

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) \leq h(r - \mu^{i-1}) \sum_{k=1}^{N-1} \parallel \bar{u}^i(k) - \bar{u}^{i-1}(k) \parallel^2 + \sigma$$

From proposition 1, we can easily derive the following corollary

Corollary 1. Let $\big(\bar{x}^i, \bar{u}^i\big)$ be a sequence generated by the algorithm. The corresponding sequence of cost values $\bar{J}(\bar{u}^i)$ are monotonically decreasing. Moreover, if $i_{max} = \infty$, the infinite sequence $\bar{J}(\bar{u}^i)$ is convergent.

Before giving the main convergence result, let us state the following corollary :

Corollary 2. Let $\big(\bar{x}^i, \bar{u}^i\big)$ be a sequence generated by the algorithm. **1)** Suppose that $i_{max} = \infty$ in order to generate an infinite sequence $\bar{u}^i \in \bar{U}$. There exists an integer $\bar{\imath}$ such that

$$\forall i \geq \bar{\imath} \quad : \quad \parallel \bar{u}^i(k) - \bar{u}^{i-1}(k) \parallel = 0 \qquad (10)$$

for all $k = 1 \dots N-1$. Moreover, if 0 is an accumulation point for the sequence μ^{i-1} then there is an accumulation point $\bar{u}^*$ of the sequence $\bar{u}^i$ that satisfies the maximum principle on the grid points. **2)** In particular, if for some $\bar{\imath}$, one has $\bar{u}^{\bar{\imath}} = \bar{u}^{\bar{\imath}-1} =: \bar{u}^*$ with $\mu^{\bar{\imath}-1} = 0$ then the control $\bar{u}^*$ satisfies the maximum principle at the grid points. **3)** In the case where i_{max} is finite, the algorithm stops after a finite number of iterations. Namely, it cannot be trapped by the loop in **(Step 3)-(Step 4)**

Finally, the following theorem is the main convergence result :

Theorem 1. Under assumption 1 and for sufficiently small $h > 0$, if $i_{max} = \infty$, then the sequence $\bar{u}^i$ produced by the above algorithm converges to some $\bar{u}^* \in \bar{U}$ that satisfies the maximum principle at the grid points.

Proof The preceding results clearly show that there exist some $i_1, i_2 \in \mathbb{N}$ such that for all $i \geq \max\{i_1, i_2\}$, one must have $\bar{J}(\bar{u}^i) > \bar{J}(\bar{u}^{i-1}) - \epsilon_J$ and $\|\bar{u}^i - \bar{u}^{i-1}\| > \epsilon_u$. Therefore, **Step 4** is never entered beyond $i \geq \max\{i_1, i_2\}$ while **Step 5** is executed at each iteration. This clearly shows that $\lim_{i\to\infty} \mu_i = 0$. This together with point 2. of corollary 2 clearly ends the proof. $\diamondsuit$

4. AN ILLUSTRATIVE EXAMPLE

In (Xu and Antsaklis, 2002), an algorithm has been proposed to solve optimal control problems on switched hybrid systems. Several examples have been used to illustrate the proposed algorithm. In particular, the following bilinear system has been considered (the other examples in (Xu and Antsaklis, 2002) concern linear systems)

$$\Sigma_1 : \begin{cases} \dot{x}_1 = -x_1 + 2x_1 v \\ \dot{x}_2 = x_2 + x_2 v \end{cases} \quad \Sigma_2 : \begin{cases} \dot{x}_1 = x_1 - 3x_1 v \\ \dot{x}_2 = 2x_2 - 2x_2 v \end{cases}$$

and

$$\Sigma_3 : \quad \begin{cases} \dot{x}_1 = 2x_1 + x_1 v \\ \dot{x}_2 = -x_2 + 3x_2 v \end{cases}$$

with the following cost function

$$J = \frac{1}{2}(x_1(T) - 2)^2 + \frac{1}{2}(x_2(T) - 2)^2 + \frac{1}{2}\int_0^T \Big[(x_1(t) - 2)^2 + (x_2(t) - 2)^2 + v^2(t)\Big] dt$$

with the initial state $x(0) = (1, 1)^T$ and $T = 3$. A sub-optimal solution of the hybrid optimal control problem above has been obtained in (Xu and Antsaklis, 2002) based on the a priori assumption of 2 switching instants. The sub-optimal J achieved by (Xu and Antsaklis, 2002) is 3.625.

The proposed algorithm has been successfully used to solve the corresponding hybrid optimal control problem using the following parameters $\epsilon_u = \epsilon_J = 0.001$, $d\mu = 0.5$; $\gamma = 1.5$, $\mu_0 = 10$. The optimal cost achieved by the algorithm is given by $\hat{J}_{\text{proposed algorithm}} = 0.3742$ to be compared to 3.625 obtained in (Xu and Antsaklis, 2002). The convergence history for example 2 is shown on figure 2. Note that at iteration 28, the condition required by corollary 2 are satisfied, namely $\mu^i = 0$; $\|\bar{u}^i - \bar{u}^{i-1}\|_\infty = 0$. Therefore,

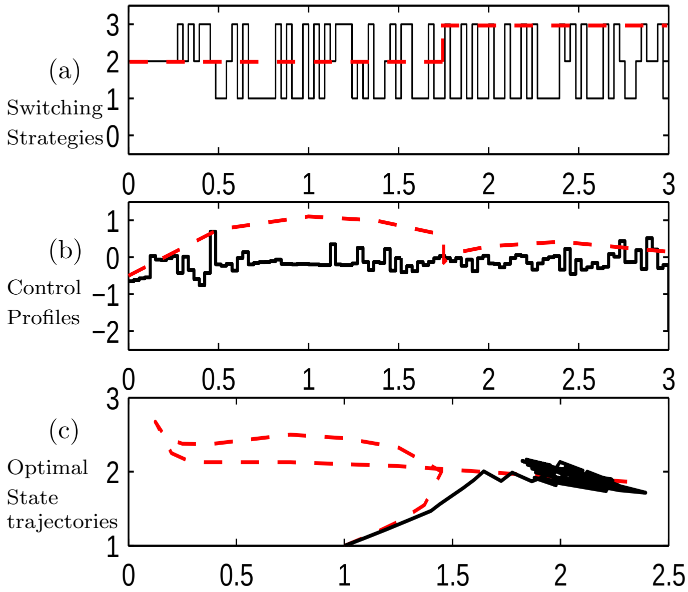

Fig. 1. Example 2.(Dotted lines - -) Solution of Xu et al. (2002) . (Solid lines –) Solution achieved by the proposed algorithm.

according to corollary 2, the sequence so obtained satisfies the maximum principle at the grid points. As for the optimal trajectories, they can be viewed on figure 1. (Experiments have been conducted using a FORTRAN 90-compiler on a PENTIUM III-600Mhz Personal Computer).

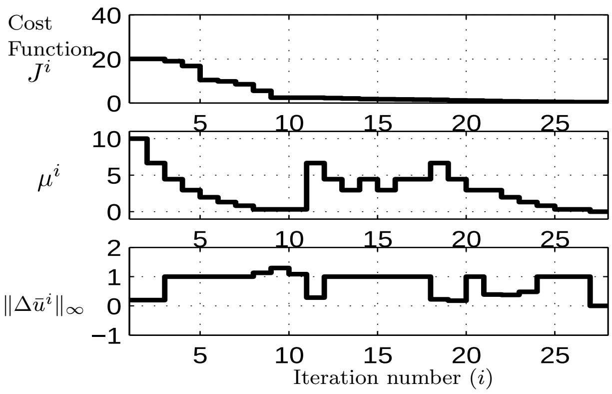

Fig. 2. Convergence results for example 2 (Execution time $\approx 1s$)

Appendix A. APPENDIX

First, the following straightforward preliminary results are needed

Proposition 2. Under assumption 1, $\forall \bar{u} \in \bar{U}$, one has for all $k = 1 \dots N$

$$\| x(t_k) - \bar{x}(k) \| \leq \rho_1(h) \; ; \; \| \lambda(t_k) - \bar{\lambda}(k) \| \leq \rho_2(h) \tag{A.1}$$

Furthermore, the approximate cost estimation error satisfies

$$|\bar{J}(\bar{u}) - J(\bar{u})| \leq \rho_3(h) \tag{A.2}$$

where the $\rho_i(.)$'s are some functions depending on the admissible set U and such that $\lim_{h\to 0} |\rho_j(h)| = 0$ $(j \in \{1, 2, 3\})$.

and by manipulating (4) using the boundedness assumption together with the well known Gronwall inequality, one obtains :

Proposition 3. Under assumption 1, there is a positive constant $M_2 > 0$ such that for all admissible control u, the solution $\lambda(.)$ of (4) in which $x(.)$ stands for the solution under the control strategy $u(.)$ satisfies the following inequality

$$\| \lambda(t) \| \leq M_2 \qquad \text{for all } t \in [t_0, t_1] \tag{A.3}$$

An immediate consequence of the above proposition is the following

Corollary 3. Under assumption 1, for all $\epsilon > 0$, there is a sufficiently small $h > 0$ such that the discretization scheme using any admissible piecewise constant control $\bar{u} \in \bar{U}$ leads to solutions $\bar{x}$, $\bar{\lambda}$ such that for all $k = 1 \dots N$:

$$\| \bar{x}(k) \| \leq M_1 + \epsilon \; ; \; \| \bar{\lambda}(k) \| \leq M_2 + \epsilon \tag{A.4}$$

<u>**Proof**</u> A straightforward consequence of (A.1), (A.3) and assumption 1 provided that $h > 0$ is sufficiently small to have $\max\big(\rho_1(h), \rho_2(h)\big) \leq \epsilon$. $\diamondsuit$

In what follows, it is assumed that the step length h has been chosen once for all and such that corollary 3 holds for $\epsilon = 1$ (for instance). Therefore, all the sequences $(\bar{x}^i, \bar{u}^i, \bar{\lambda}^i, t_k)$ generated by the above algorithm lie in the compact set

$$\mathcal{S} := \bar{X} \times \bar{U} \times \bar{\Lambda} \times [t_0, t_f] \subset \Re^{2Nn+(N-1)m+1} \tag{A.5}$$

where $\bar{X} := \bar{B}(0, M_1 + 1) \times \dots \times \bar{B}(0, M_1 + 1) \subset \Re^{Nn}$ and $\bar{\Lambda} := \bar{B}(0, M_2 + 1) \times \dots \times \bar{B}(0, M_2 + 1) \subset \Re^{Nn}$.

A.1 Proof of proposition 1

Throughout the proof, the notation $F^k(X_1, X_2, \dots)$ is used to shortly designate $F(X_1(k), X_2(k), \dots)$. We have by definition (9) $\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) = h \sum_{k=1}^{N-1} \Big[L^k(\bar{x}^i, \bar{u}^i, t) - L^k(\bar{x}^{i-1}, \bar{u}^{i-1}, t)\Big]$. Using the definition of the Hamiltonian, one can write

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) = h\sum_{k=1}^{N-1}\Big[H^k(\bar{x}^i, \bar{u}^i, \bar{\lambda}^{i-1}, t) - H^k(\bar{x}^i, \bar{u}^{i-1}, \bar{\lambda}^{i-1}, t) + H^k(\bar{x}^i, \bar{u}^{i-1}, \bar{\lambda}^{i-1}, t) - -H^k(\bar{x}^{i-1}, \bar{u}^{i-1}, \bar{\lambda}^{i-1}, t) - \Big(f^k(\bar{x}^i, \bar{u}^i, t) - -f^T(\bar{x}^{i-1}, \bar{u}^{i-1}, t)\Big)\bar{\lambda}^{i-1}\Big] \tag{A.6}$$

we shall successively examine each group of two consecutive terms in (A.6). **For the first** one we have by definition of $\bar{u}^i$ (see step 3 of the algorithm)

$$H^k(\bar{x}^i, \bar{u}^i, \bar{\lambda}^{i-1}, t) - H^k(\bar{x}^i, \bar{u}^{i-1}, \bar{\lambda}^{i-1}, t) \leq -\mu^{i-1} \parallel \bar{u}^i(k) - \bar{u}^{i-1}(k) \parallel^2 \tag{A.7}$$

as for **the second group**, we can write (using $\delta\bar{x}^i$ to denote $\bar{x}^i - \bar{x}^{i-1}$ and $r_1 = \sup_{z\in\mathcal{S}} \parallel H_{xx}(z) \parallel$)

$$H^k(\bar{x}^i, \bar{u}^{i-1}, \bar{\lambda}^{i-1}, t) - H^k(\bar{x}^{i-1}, \bar{u}^{i-1}, \bar{\lambda}^{i-1}, t) \leq H^k_x(\bar{x}^{i-1}, \bar{u}^{i-1}, \bar{\lambda}^{i-1}, t)\delta\bar{x}^i(k) + r_1 \parallel \delta\bar{x}^i(k) \parallel^2 \tag{A.8}$$

using (A.7) and (A.8) into (A.6), we obtain

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) \leq h\Big[-\mu^{i-1}\sum_{k=1}^{N-1} \parallel \bar{u}^i(k) - \bar{u}^{i-1}(k) \parallel^2 + r_1\sum_{k=1}^{N-1} \parallel \delta\bar{x}^i(k) \parallel^2\Big] + I^i(h) \tag{A.9}$$

where $I^i(h) = -\sum_{k=1}^{N-1}\Big[\Delta\big(\delta^T\bar{x}^i(k)\bar{\lambda}^{i-1}(k)\big) + o(h)\Big]$ since $h = (t_f - t_0)/(N-1)$ then $\sum_{k=1}^{N-1} o(h) = O(h)$ and hence

$I^i(h) = \delta^T\bar{x}^i(1)\bar{\lambda}^{i-1}(1) - \delta^T\bar{x}^i(N)\bar{\lambda}^{i-1}(N) + O(h)$

however, according to the boundary conditions on x and λ (respected by the algorithm), we have $\bar{\lambda}^{i-1}(N) = 0$ and $\delta\bar{x}^i(1) = 0$ for all i, therefore $I^i(h) = O(h)$. this with (A.9) gives

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) \leq h\Big[-\mu^{i-1}\sum_{k=1}^{N-1} \parallel \bar{u}^i(k) - \bar{u}^{i-1}(k) \parallel^2 + r_1\sum_{k=1}^{N-1} \parallel \delta\bar{x}^i(k) \parallel^2\Big] + O(h) \tag{A.10}$$

We shall prove hereafter that there is positive constant $r_2 > 0$ such that

$$\sum_{k=1}^{N-1} \parallel \delta\bar{x}^i(k) \parallel^2 \leq r_2.\sum_{k=1}^{N-1} \parallel \bar{u}^i(k) - \bar{u}^{i-1}(k) \parallel^2 \tag{A.11}$$

for this, let us write (7) in the following compact form

$$\bar{x}(k+1) := \bar{x}(k) + hF^{(k)}(\bar{x}(k), \bar{u}(k), t_k, t_{k+1}) \tag{A.12}$$

where $F^{(k)}$ is clearly defined by identification with (7). Note also that according to the regularity assumptions, $F^{(k)}$ is C^2 in its arguments. From (A.12) we can write

$$\delta\bar{x}^i(k+1) = h\Big[F^{(k)}_x(\theta^i(k), w^i(k), t_k, t_{k+1})\delta\bar{x}^i(k) + +F^{(k)}_u(\theta^i(k), w^i(k), t_k, t_{k+1})[\bar{u}^i(k) - \bar{u}^{i-1}(k)]\Big] \tag{A.13}$$

where $\theta^i(k) \in \bar{X}$ and $w^i(k)$ belongs to the convex hull $[\bar{U}]$ of $\bar{U}$. Therefore, using γ_1 and γ_2 defined by

$$\gamma_1 = \sup_{(\theta,w,\tau_1,\tau_2)\in\bar{X}\times[\bar{U}]\times[t_0,t_f]^2} \max_k \parallel F^{(k)}_x(\theta, w, \tau_1, \tau_2) \parallel$$

$$\gamma_2 = \sup_{(\theta,w,\tau_1,\tau_2)\in\bar{X}\times[\bar{U}]\times[t_0,t_f]^2} \max_{k=1} \parallel F^{(k)}_u(\theta, w, \tau_1, \tau_2) \parallel$$

equation (A.13) can be rewritten as follows

$$\parallel \delta\bar{x}^i(k+1) \parallel \leq h\Big[\gamma_1 \parallel \delta\bar{x}^i(k) \parallel + \gamma_2 \parallel \bar{u}^i(k) - \bar{u}^{i-1}(k) \parallel\Big]$$

which gives after some manipulations, one obtains for all $k \geq 2$

$$\parallel \delta\bar{x}^i(k) \parallel \leq h\gamma_2\sum_{j=2}^{k}\gamma_1^{j-2} \parallel \bar{u}^i(j-1) - \bar{u}^{i-1}(j-1) \parallel \leq$$

$$\leq h\gamma_2\sum_{j=1}^{k-1}\gamma_1^{j-1} \parallel \bar{u}^i(j) - \bar{u}^{i-1}(j) \parallel$$

now using $\bar{\gamma}_1 := \max_{j=1\ldots N-1}\{\gamma_1^j\}$, we obtain for all $k \geq 2$:

$$\parallel \delta\bar{x}^i(k) \parallel \leq h\gamma_2\bar{\gamma}_1\sum_{j=1}^{N-1} \parallel \bar{u}^i(j) - \bar{u}^{i-1}(j) \parallel \tag{A.14}$$

therefore, using Cauchy-Schawrtz inequality and summing (A.14) for $k = 1$ to $k = N-1$

$$\sum_{k=1}^{N-1} \parallel \delta\bar{x}^i(k) \parallel^2 \leq h^2(\gamma_2\bar{\gamma}_1)^2(N-1)\sum_{j=1}^{N-1} \parallel \bar{u}^i(j) - \bar{u}^{i-1}(j) \parallel^2 = \frac{(t_f - t_0)^2}{N-1}(\gamma_2\bar{\gamma}_1)^2\sum_{k=1}^{N-1} \parallel \bar{u}^i(k) - \bar{u}^{i-1}(k) \parallel^2$$

which is nothing else that (A.11) with $r_2 = \dfrac{(t_f - t_0)^2}{N-1}(\gamma_2\bar{\gamma}_1)^2$. Hence, (A.10) can be written in the form (taking $r := r_1 r_2$)

$$\bar{J}(\bar{u}^i) - \bar{J}(\bar{u}^{i-1}) \leq$$
$$h(r - \mu^{i-1}) \sum_{k=1}^{N-1} \| \bar{u}^i(k) - \bar{u}^{i-1}(k) \|^2 + O(h)$$

which gives the result whenever h is sufficiently small. This ends the proof of proposition 1 $\diamondsuit$

A.2 Proof of Corollary 1

Consider the sequence $(\bar{u}^i)$, for each i, two situations are possible: 1) $\| \bar{u}^i - \bar{u}^{i-1} \|_\infty = 0$, in this case, $\bar{J}(\bar{u}^i) = \bar{J}(\bar{u}^{i-1})$. 2) $\| \bar{u}^i - \bar{u}^{i-1} \|_\infty > 0$, in this case, according to proposition 1, it is always possible to find sufficiently high μ^{i-1} (obtained after successive application of $\mu^{i-1} = \mu^{i-1} + d\mu$ of **step 4**) such that the condition $(\bar{J}(\bar{u}^i) \leq \bar{J}(\bar{u}^{i-1}) - \epsilon_J)$ necessary to leave step **step 4** is satisfied. Therefore, in this case, we have $\bar{J}(\bar{u}^i) < \bar{J}(\bar{u}^{i-1})$. Therefore, $\bar{J}(\bar{u}^i)$ can only decrease. Now, when $i_{max} = \infty$, the algorithm generates an infinite decreasing sequence $\{\bar{J}(\bar{u}^i)\}$ that it is bounded below hence convergent. $\diamondsuit$

A.3 Proof of Corollary 2

In order to prove 1), it is sufficient to prove that there is only a finite number of integer i such that $\| \bar{u}^i - \bar{u}^{i-1} \|_\infty > 0$. For, suppose that this is not true, then there will be an infinite subsequence i_j of indices such that $\| \bar{u}^{i_j} - \bar{u}^{i_j-1} \|_\infty > 0$. According to the proof of corollary 1, however, this subsequence must satisfy $\bar{J}(\bar{u}^{i_j}) \leq \bar{J}(\bar{u}^{i_j-1}) - \epsilon_J$ and $J(\bar{u}^{i_j})$ tends to $-\infty$ when j tends to ∞. This is impossible because the cost function values are lower bounded. By definition of $\bar{u}^i$ (see **Step 3**), we can write for all $k = 1 \ldots N-1$ and all $\bar{u} \in \bar{U}$ that $H^k(\bar{x}^i, \bar{u}^i, \bar{\lambda}^{i-1}, t) + \mu^{i-1} \| \bar{u}^i(k) - \bar{u}^{i-1}(k) \|^2 \leq H^k(\bar{x}^i, \bar{u}, \bar{\lambda}^{i-1}, t) + \mu^{i-1} \| \bar{u}(k) - \bar{u}^{i-1}(k) \|^2$. Now taking the subsequence i_j such that $\lim_{j\to\infty} \mu^{i_j-1} = 0$, and taking the limit ends the proof of 1). To prove 2) it is sufficient to remark that if $\bar{u}^{\bar{\imath}} = \bar{u}^{\bar{\imath}-1} =: \bar{u}^*$ with $\mu^{\bar{\imath}-1} = 0$ then the sequence $\bar{u}^i$ for $i \geq \bar{\imath}$ becomes constant and $\bar{u}^*$ is a trivial accumulation point of the control sequence. The result follows from point 1). To prove 3), it is sufficient to note that, using the result of 1), the algorithm stops when i satisfies $i \geq \max\{i_{max}, \bar{\imath}\}$ (see the stop condition of **Step 6**). $\diamondsuit$

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