

## TOWARDS THE CONSTRUCTION OF A ROBUST OPTIMAL OBSERVER

Luis Antonio Calvillo-Corona and Mazen Alamir

*Laboratoire d'Automatique de Grenoble/CNRS UMR  
5528, LAG-ENSIEG BP 46, Domaine Universitaire/Saint  
Martin d'Hères, France  
email : calvillo@lag.ensieg.inpg.fr  
Mazen.Alamir@inpg.fr*

**Abstract:** In this paper, an algorithm for the construction of nonlinear optimal observers is proposed. The key feature is the definition of an elementary problem, which is a scalar optimal control problem. The resolution of this problem is done in two stages: 1) Solution of a scalar Hamilton-Jacobi equation to find an optimal cost function of the unknown initial state and 2) Choosing the initial state to minimize this function.

**Keywords:** robust optimal observers, estimation algorithms, Hamilton-Jacobi equation, heuristic.

### 1. INTRODUCTION

In many practical situations it is not possible to accede to all the state variables of a system to apply the state feedback. This is due to the technological limitations that exist to measure the states directly or because the measurement devices (sensors) are scants because of economical restrictions. Then, to be able to apply the state feedback, an estimation of the vector of states must be obtained.

For the linear case, the solution for the observation problem can be obtained by the well known Luenberger observer (Luenberger, 1971). But for the case of nonlinear systems, there is no systematic method to solve the general observation problem. Also, the methods to construct nonlinear observers present at least one of the next drawbacks: restrictive structural conditions to be satisfied, model's errors and disturbances that are not explicitly taken into account, heavy calculations to be performed, etc. It is clear that a survey of existing works is beyond the scope of this paper, related surveys can be found in (Alamir, 1999; Be-

sançon, 1996; Misawa and Hedrick, 1999; Walcott et al., 1987; Gow-Bin et al., 1997)

In this paper, interest is focussed on a new method to construct a robust optimal nonlinear observer. The method consists first in dividing the main problem into  $n$  scalar ones, each scalar problem is called an "elementary problem" (**EP**). Then in solving each scalar optimization problem using a scalar Hamilton-Jacobi equation's solver. The solutions of the different scalar problems is then "analysed" to define a new set of scalar problems to the next iteration. Iterations continue until all these solutions are mutually compatible.

While multi-dimensional Hamilton-Jacobi equations (**HJE**) are quite difficult to solve in general (Sage, 1977), scalar (**HJE**) can nowadays benefit of powerful numerical solvers. The algorithm proposed in this paper is a first step towards the construction of nonlinear observation scheme that is independent of any structural properties with the ability to handle bounded noises and modelling uncertainties.

This work is rather exploratory, it may be viewed as a first step towards the construction of merging-

horizon like observer that is suitable to explicitly handle uncertain and perturbed systems.

This paper is organized as follows: First, some notations are introduced and the problem is formulated in section 2. Then the principle of the algorithm is presented in section 3. The notion of elementary problem and its resolution are exposed in section 4. Finally, in section 5 an illustrative example is proposed.

## 2. THE OBSERVER PROBLEM

Consider nonlinear systems given by:

$$\dot{x} = f(x, u, \delta, t) \quad ; \quad y = h(x, \delta, t) \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$ ,  $u \in \mathbb{R}^m$  and  $\delta \in \mathbb{R}^q$  stand for state, measured output, control and disturbance/uncertainty vectors respectively. The disturbance/uncertainty vector  $\delta$  is supposed to verify the following constraint:

$$\forall t \geq 0 \quad ; \quad |\delta_i(t)| \leq \delta_i^{max} \quad ; \quad i = 1, \dots, q \quad (2)$$

where  $(\delta_i^{max})_{i=1}^q$  is an a priori known sequence with upper limits.

Also consider some horizon of observation  $T > 0$  and a sampling period  $s > 0$  such that  $(N - 1)s = T$  where  $N \in \mathbb{N}$  is a strictly positive integer.

### 2.1 The problem formulation

The objective is to propose an effective method that finds a good suboptimal solution of the following observation problem:

*The measures of  $y$  and  $u$  being acquired with a sampling rate  $s$ , propose an algorithm that approximately solve the optimal filtering problem defined by the following criterion:*

$$\hat{x}(t), \hat{\delta}(t) \leftarrow \min_{\tilde{x}(\cdot), \tilde{\delta}(\cdot)} \sum_{i=1}^N \|y(t - is) - \hat{y}(t - is)\|^2$$

$$\hat{y} := h(\tilde{x}(\tau), \tilde{\delta}(\tau), \tau) \quad (3)$$

under the constraints (1) and (2). Using the notation

$$\bar{Y}(t) := (y(t - Ns), \dots, y(t - s))^T$$

the observer dynamics is given by

$$\bar{Y}(t) = A\bar{Y}(t - s) + B y(t - s)$$

$$\left( \hat{x}(t), \hat{\delta}(t) \right) = \mathcal{F}(\bar{Y}(t), t)$$

where  $\mathcal{F}(\bar{Y}(t), t)$  is the optimal solution of (3) while  $A$  and  $B$  are classical finite-memory related matrices.

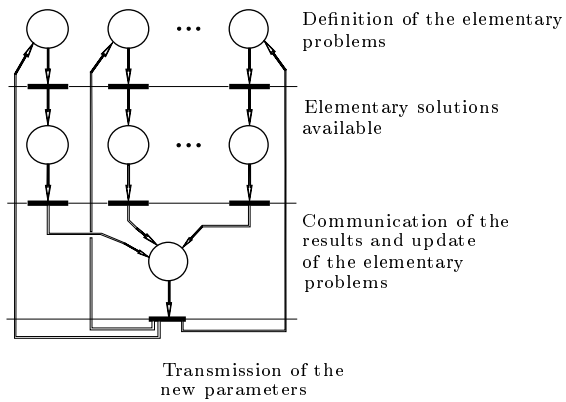


Fig. 1. The structure of the algorithm

## 3. THE PRINCIPLE OF THE ALGORITHM

In this section the principle of the algorithm is stated, and a simple example is given to clarify it. The scheme of Figure 1 summarizes the principle using Petri nets diagram.

The design of the proposed resolution algorithm passes by the definition of an elementary problem (**EP**). The resolution of such (**EP**) is the role of an agent. The solution of the original problem is done by the synchronized communication between several agents solving each one an (**EP**). Thus, the resolution of all (**EP**) can be done in parallel. At the end of this phase each agent has the solution of its own problem. Then a single processing phase of the results of each agent is performed to redefine each new (**EP**) according to the results obtained by the set of the agents. A parallel phase of calculation can then start again.

To illustrate the principle described above, consider the following modified Van der Pol system:

$$\dot{x}_1 = x_2 + u + v(t) \quad (4)$$

$$\dot{x}_2 = -9x_1 + \mu(t)(1 - x_1^2)x_2 \quad (5)$$

$$y = x_1 \quad (6)$$

where  $v$  is some unmeasured external signal. As the measures of the output  $x_1 = y$  are obtained on a given interval, an optimal control problem can be defined in which (4) is the dynamic system,  $x_1$  is the state,  $(v, x_2)$  is the control vector and the measured signal is the reference to be tracked. The resulting optimal trajectory of the state  $x_1$  and the control  $x_2$ , denoted by  $x_1^{(k,1)}$  and  $x_2^{(k,1)}$  are obtained.

In the same way, the optimal solutions  $x_j^{(k-1,i)}$  are used to define reference signals for a scalar optimal control problem in which (5) is the dynamical system,  $x_2$  is the state,  $x_1$  is the control and  $\mu$  is a well-known signal. The resulting optimal solutions are denoted by  $x_1^{(k,2)}$  and  $x_2^{(k,2)}$ .

As it is noticed, two trajectories for  $x_1$  will be obtained and two for  $x_2$ , one coming from each (**EP**) agent solver, a process of the procured

results will therefore be made in order to obtain a single current estimation (at iteration  $k$ ) for  $x_1$  and  $x_2$ . After this, the iteration can be continued.

#### 4. THE ELEMENTARY PROBLEM

The **(EP)** is an optimal control problem with unknown initial state of a dynamic system with only one state, that is, a dynamic system given by:

$$\dot{\xi} = F(\xi, x^c, \nu, \tau) \quad ; \quad \tau \in [t - T, t] \quad (7)$$

where

- $\xi \in \mathfrak{R}$  is the state.
- $x^c(\cdot)$  is a signal whose evolution is known on  $[t - T, t]$ .
- $\nu$  is a control vector satisfying a constraint of the type  $\nu(\tau) \in V$ .

The problem then consists in solving for system (7) the optimal control problem with unknown initial state according to:

$$\min_{\xi(\cdot); \nu(\cdot)} \int_{t-T}^t q(\xi(\tau) - \xi^{ref}(\tau))^2 + r \|\nu(\tau) - \nu^{ref}(\tau)\|^2 \quad (8)$$

in which  $\xi^{ref}(\cdot)$  and  $\nu^{ref}(\cdot)$  are trajectories of reference given a priori.

##### 4.1 General resolution of the elementary problem

As it has been mentioned in the previous part of the section, an **(EP)** is an optimal control problem with the following characteristics:

- The system is variant in the time.
- The optimization horizon is finite.
- The initial state is unknown.

In order to solve this problem, it will be proceeded in two stages:

- (1) The **(EP)** represented by equations (7)-(8) is transformed into a scalar Hamilton-Jacobi equation **(HJE)**. This equation is solved using existing and effective codes of calculation. In addition, this solution gives the values of the optimal solutions corresponding to all possible initial states. In effect, the solution of these equations leads to a function  $V(0, \xi)$  that represents the optimal value of the criterion in function of the initial state.
- (2) The initial state then is chosen by minimizing  $V(0, \xi)$ , i.e.,  $\hat{\xi}(t - T) = \arg\{\min_{\xi} V(0, \xi)\}$ . It is clear that the initial state  $\hat{\xi}(\cdot)$  and the control signal  $\hat{\nu}(\cdot)$  are obtained by integration.

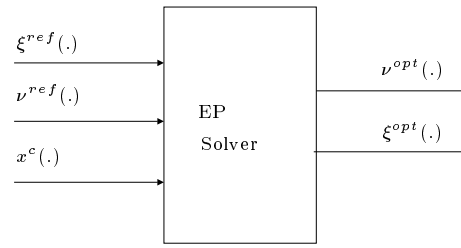


Fig. 2. Inputs and outputs of the elementary problem resolution module.

Summarizing, this procedure consists in dividing the unknowns into two groups. The initial condition of the state in one part and the control in another one. The optimal control is first found in function of the initial state, then it is chosen immediately to minimize the initial global criterion. The formalism of Hamilton-Jacobi for the search of laws of control by state feedback minimizing a given criterion is briefly recalled. The formalism (Isaacs, 1954–1956) stipulates that if there exists a  $C^1$  function  $V(\tau, \xi)$  that satisfies the following partial differential equations (with terminal condition  $V(T, \xi) = 0$ ):

$$V_{\tau}(\tau, \xi) + \bar{H}(\tau, \xi, V_{\xi}(\tau, \xi)) = 0 \quad (9)$$

in which

$$\bar{H}(\tau, \xi, V_{\xi}) = H(\tau, \xi, x^c, \hat{\nu}(\tau, \xi, V_{\xi}), V_{\xi})$$

$$H(\tau, \xi, x^c, \nu, \lambda) = L(\tau, \xi, x^c, \nu) + \lambda^T F(\tau, \xi, x^c, \nu)$$

$$\hat{\nu}(\tau, \xi, V_{\xi}) = \arg\{\min_{\nu \in V} H(\tau, \xi, x^c, \nu, V_{\xi})\}$$

then, to solve the **(EP)** defined by equations (7)-(8) one may integrate the scalar partial differential equation (9) backward in time to obtain the function  $V(0, \xi)$ .

A resolution module of an **(EP)** is presented as indicated on Figure 2.

#### 5. ILLUSTRATIVE EXAMPLE

The methodology described above will be applied to the modified Van der Pol system (4)-(6).

Note that with respect to the general form (1),  $\delta = v$ ,  $u(\cdot)$  and  $\mu(\cdot)$  are known functions of time, that is,  $x^c = [u \quad \mu]^T$ .

It is supposed that measures have been acquired since at least  $T$  seconds. Thus, it makes use of the  $N$  last samples of measurement of the output  $y$  and of the control input  $u$ .

$$Y_m(t) := [y_m(t - Ns), \dots, y_m(t - s)]^T$$

$$U(t) := [u(t - Ns), \dots, u(t - s)]^T$$

It is supposed in addition that an initial estimate of the trajectories of all the signals is available, more precisely, one has  $x^{(0)}(\tau)$  and  $\delta^{(0)}(\tau)$ .

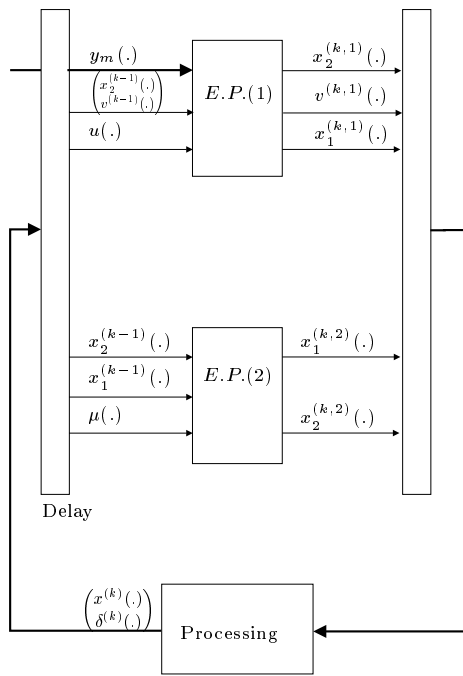


Fig. 3. Elementary problems for the illustrative example

This information is summarized by the values of these signals at different instants of sampling, the corresponding function is then obtained by simple cubic spline interpolation.

The diagram of Figure 3 makes it possible to understand the logic of the proposed algorithm in the case of system (4)-(6).

### 5.1 Explanation of Figure 3

- On the diagram of the Figure 3, the notations  $\eta^{(k,1)}$  and  $\eta^{(k,2)}$  represent the results (concerning a variable  $\eta$ ) of the resolution at iteration  $k$  of the **(EP 1)** and **(EP 2)** respectively.
- In the **(EP 1)**,  $x_1$  is the state,  $(x_2, v)^T$  is the control,  $u$  is a known signal. The reference signals are defined starting from the solution of the **(EPs)** at the preceding iteration (after appropriate processing). With respect to the definition of the **(EP)** [see (7)-(8)], the following correspondences hold

$$\xi \leftrightarrow x_1 ; x^c \leftrightarrow u ; \nu \leftrightarrow \begin{pmatrix} x_2 \\ v \end{pmatrix}$$

$$\xi^{ref}(\cdot) \leftrightarrow y_m(\cdot) ; \nu^{ref} \leftrightarrow \begin{pmatrix} x_2^{(k-1)} \\ v^{(k-1)} \end{pmatrix}$$

- In the **(EP 2)**,  $x_2$  is the state,  $x_1$  is the control,  $\mu$  is a known signal. With respect to the definition of the **(EP)**, the following correspondences hold

$$\xi \leftrightarrow x_2 ; x^c \leftrightarrow \mu ; \nu \leftrightarrow x_1$$

$$\xi^{ref}(\cdot) \leftrightarrow x_2^{(k-1)}(\cdot) ; \nu^{ref} \leftrightarrow x_1^{(k-1)}$$

- The processing block is supposed to define the values finally selected at the end of the iteration  $k$  insofar as certain values are obtained at the same time by the resolution module for problem 1 and problem 2. An example of such processing can be quite simply an average. Weighted averages with judiciously selected coefficients can be considered later. Thus, a way of carrying out the processing is the following one:

$$x_i^{(k)} := \frac{x_i^{(k,1)} + x_i^{(k,2)}}{2} ; \quad i = 1, 2 \quad (10)$$

- Finally, at the end of the iteration  $k$ , the estimate of the state is given by  $x^{(k)}(t)$

As it has been set out previously, the main idea of this algorithm is to solve each equation of the system like independent systems but using coupled reference signals in order to force the final solution to be compatible with all system's equations.

### 5.2 Results

The data used during the simulation are the following:

- The initial conditions are  $x(0) = [0.5, -0.3]^T$
- $t_0 = 0, t_f = 10$
- $q = [10, 1 \times 10^{-3}]^T, r = \begin{pmatrix} 1 \times 10^{-3} & 5 \times 10^{-4} \\ 10 & 0 \end{pmatrix}$  are the penalty coefficients.
- The known signals are  $u = \sin(t), \mu = 0.1 \sin(t)$

At iteration 1, the algorithm begins with the initialization of the Van der Pol system (4)-(6), which is integrated with erroneous initial state for  $x_2$ , this initial value is 0, in addition to this error, the equation (5) is divided by 0.5, to obtain initial reference signals. The external  $v$  used is given by

$$v = \begin{cases} 0, & t \leq 0.1t_f \\ \left(\frac{t - 0.1t_f}{0.1t_f}\right)0.5, & 0.1t_f < t \leq 0.2t_f \\ 0.5, & 0.2t_f < t \end{cases} \quad (11)$$

On Figure 4 the solutions of the **(EP1)** are shown. In the upper graphic  $\xi^{ref}$  represents the true state  $x_1$  which is the measured output  $y$ . As it can be noted, the estimation of the trajectory of the state  $x_1$  is already obtained,  $\hat{\xi} = \xi^{ref}$ , since the reference is the measured output  $x_1 = y$ . In the middle graphic it can be noted that the estimation of  $x_2$ , namely  $\hat{\nu}_1$ , is far from the true state  $x_2$ , because of the given reference  $\nu_1^{ref}$ . In the lower

graphic the behaviour of the estimation, namely  $\hat{\nu}_2$ , of the external signal  $v$ , with reference  $\nu_2^{ref} = 0$  is presented.

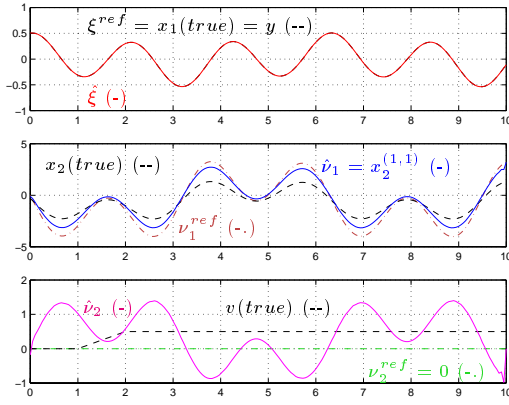


Fig. 4. Elementary Problem 1 at iteration 1

In Figure 5 the solutions for the **(EP2)** are shown. In the upper graphic  $\nu^{ref}$  represents the true state  $x_1$  which is the measured output  $y$ . As it can be noted, the estimation of the trajectory of the state  $x_1$  is already obtained. In the lower graphic it can be noted that the estimation, namely  $\hat{\xi}$ , has not yet converged to the true state  $x_2$ , because the given reference, namely  $\xi^{ref}$ , is the initial trajectory with an erroneous initial state  $x_2^{(0)}$ .

Note that the estimation for  $x_1$  is already obtained for both **(EP)**, then, for the other figures this graphics will not be presented.

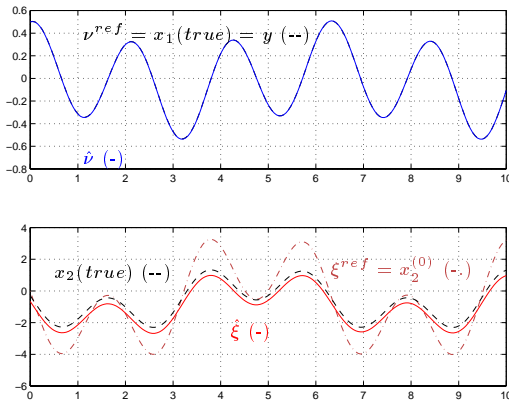


Fig. 5. Elementary Problem 2 at iteration 1

The final solutions for  $x_1$  and  $x_2$  are obtained from the processing phase at the end of iteration 1. These new solutions will be used in the next iteration.

In the upper graphic of Figure 6 it can be noted that the estimation of  $\hat{\nu}_1$ , is closer to the true state  $x_2$ , in spite of the reference  $\nu_1^{ref}$  and this is because of the results obtained from the processing phase. In the lower graphic of this figure, it can also be noted an improvement in the behaviour

of the trajectory  $\hat{\nu}_2$  with respect to the preceding iteration.

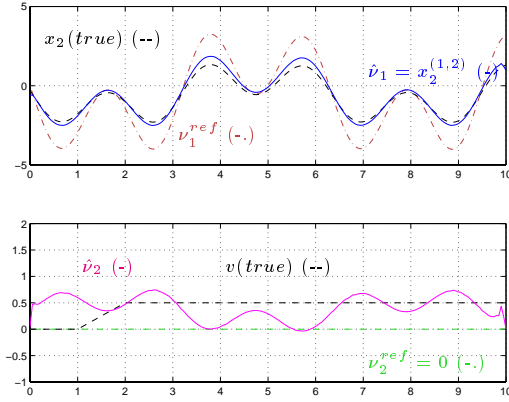


Fig. 6. Elementary Problem 1 at iteration 2

The results obtained for the **(EP2)** at iteration 2 are very closer to those obtained at iteration 1, but in this case, there is no reference signal  $\xi^{ref}$ . [see Figure 7 and Figure 5]

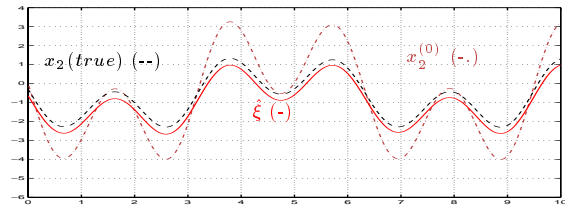


Fig. 7. Elementary Problem 2 at iteration 2

It can be noted in the upper graphic of Figure 8 that the estimation of the trajectory of the state  $x_2$ , namely  $\hat{\nu}_1$ , is already obtained, and in the bottom graphic that an estimation of  $v$  is not achieved, but a trajectory that globally reflects the behaviour of  $v$  is obtained.

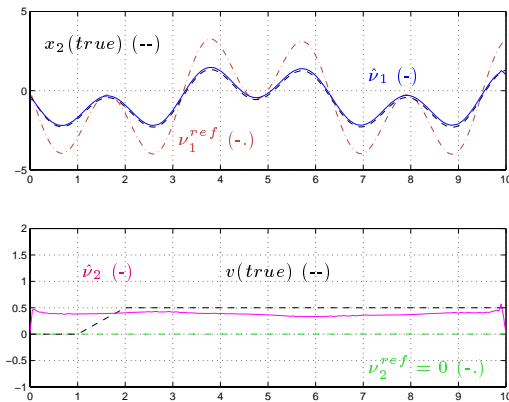


Fig. 8. Elementary Problem 1 at iteration 5

In Figure 9 the final solutions of the states trajectories after the processing phase at iteration 2 and iteration 5 are shown. In the lower graphic of this figure, it can be noted that the estimation

of the trajectory of the state  $x_2$  has been practically achieved in only two iterations, although the initial trajectory was far from the real trajectory.

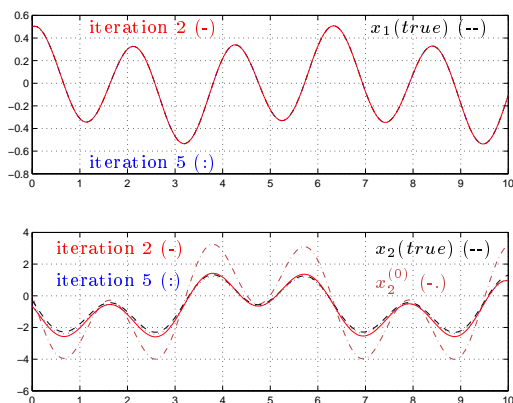


Fig. 9. The trajectory of the states after the processing phase at different iterations

The algorithm stops when the maximum value of the absolute value of the error between the solution of the processing phase at iteration  $k$  and the solution of the processing phase at iteration  $(k-1)$  of the trajectory of the state  $x_2$  is less or equal to a given  $\epsilon$ . That is

$$\max_t |x_2^k(t) - x_2^{(k-1)}(t)| \leq \epsilon \quad ; \quad \epsilon = 1 \times 10^{-4}$$

[see Figure 10 ]

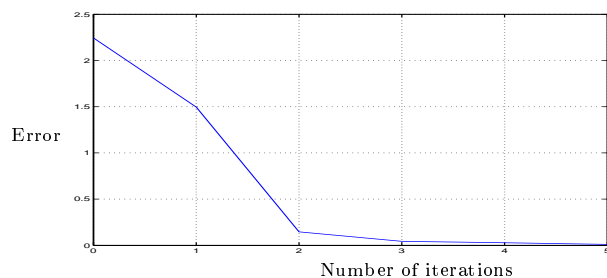


Fig. 10. The error at each iteration

## 6. CONCLUSION

In this paper, an algorithm towards the construction of robust optimal observers is presented. The concept of **(EP)** is introduced as well as the way to solve it.

This methodology presents the advantage that it is not necessarily to know the structural properties of the system to design the observer. Also, this methodology allows to separate the original problem in scalar problems, that means that systems of great order can be solved with relative facility, in addition to this, parallel implementation is straightforward and may accelerate the resolution of the global observation problem.

Furthermore, even for uncertain/perturbed systems for which observability is not easy to qualify, the proposed algorithm may be used as a study

tool enabling to show whether several signals configuration may lead to the same output under certain circumstances.

There are three main issues to be considered in later works in order for the proposed approach to become really operational: Convergence properties, weighting parameters tuning and execution time.

## REFERENCES

- M. Alamir. Optimization based non-linear observers revisited. *International Journal of Control*, **72**:1204–1217, 1999.
- G. Besançon. *Contributions à l'étude et à l'observation des systèmes non linéaires avec recours au calcul formel*. PhD thesis, *Thèse de Doctorat, INPG, France*, 1996.
- W. Gow-Bin, P. Sheng-Shiou, and H. Hsiao-Ping. A sliding observer for nonlinear process control. *Chemical Engineering Science*, **52**:787–805, 1997.
- R. Isaacs. Differential games, i, ii, iii, iv. Technical Report **RM-1391**, **1399**, **1411**, **1468**, *RAND Corporation Research Memorandum*, 1954–1956.
- D. G. Luenberger. An introduction to observers. *I.E.E.E. Transactions on Automatic Control*, **16**:592–602, 1971.
- E.A. Misawa and J.K. Hedrick. Nonlinear observers -a-state-of-the-art survey. *Journal of Dynamical Systems Measurement and Control*, **72**:1204–1217, 1999.
- Andrew P. Sage. *Optimum Systems Control*. Prentice Hall, 1977.
- B. Walcott, M. Coreless, and S. Zak. Comparative study of nonlinear observation techniques. *International Journal of Control*, **45**:2109–2132, 1987.