A New Contraction-Based NMPC Formulation Without Stability-Related Terminal Constraints

Mazen Alamir *

* *CNRS, University of Grenoble Alpes*
Email: mazen.alamir@grenoble-inp.fr

Abstract: Contraction-Based Nonlinear Model Predictive Control (NMPC) formulations are attractive because of the generally short prediction horizons they require and the needless use of terminal set computation that are commonly necessary to guarantee stability. However, the inclusion of the contraction constraint in the definition of the underlying optimization problem often leads to non standard features such as the need for multi-step open-loop application of control sequences or the use of multi-step memorization of the contraction level that may induce unfeasibility in presence of unexpected disturbance. This paper proposes a new formulation of contraction-based NMPC in which no contraction constraint is explicitly involved. Convergence of the resulting closed-loop behavior is proved under mild assumptions.

Keywords: Model Predictive Control, Nonlinear Systems, Short Prediction Horizon.

1. INTRODUCTION

Provable closed-loop stability in the majority of NMPC formulations results from the use of terminal constraints on the state. In the early formulations (Keerthi and Gilbert, 1988; Mayne and Michalska, 1990), stringent equality constraint on the state is used. Then relaxations were introduced through the combined use of terminal set inclusion and appropriate terminal penalty. The many different ways to choose these two items were unified in (Mayne et al., 2000) where it has been shown that the terminal set should be controlled-invariant under some *local* feedback control that makes the terminal penalty a control-Lyapunov function. This pair of terminal set and terminal penalty function are the most often computed based on Linear Quadratic Regulator (LQR) design if the linearized system around the targeted state is stablilizable. Otherwise, the recently proposed scheme (Lazar and Spinu, 2015) can be used for purely nonlinear systems through extensive use of the finite step Lyapunov function paradigm. Regardless of the way this pair is computed, the feasibility of the associated terminal constraint generally needs long prediction horizons to be used in the MPC formulation. Moreover, the presence of this constraint makes the computation of the optimal solution a difficult task. This may explain why many practitioners *confess* never including such stability-related constraints in their formulations even in applications where the latter is almost dedicated to stabilization.

On the other hand, it has been shown quite early (Alamir and Bornard, 1995) that provable stability can be obtained without terminal stability-related constraint by using *sufficiently long* prediction horizon (Grimm et al., 2005; Jadbabaie and Hauser, 2005). More recent results followed, [see (Grüne et al., 2010; Grüne and Pannek, 2011; Boccia et al., 2014) and the references therein] where deeper analysis is obtained regarding this fact. However, the underlying argument remained that with sufficiently long prediction horizon, the optimal decisions necessarily lead to open-loop trajectories with terminal appropriate properties.

For obvious computational reasons, one might be interested in formulations involving short prediction horizons and no stability-related terminal constraints. Following the previous argumentation, this might appear paradoxical. This is precisely where contractive formulations enter into picture. Indeed, the contraction property for a controlled system is the systematic ability to find a control sequence $\boldsymbol{u}$ that steers the state of the system from its current value x_k to a new state x_{k+N} where the value of some positive definite function W is contracted by some ratio $\gamma \in (0,1)$, namely $W(x_{k+N}) \leq \gamma W(x_k)$. Now when $N = 1$, this property is satisfied only for difficult-to-find standard controlled-Lyapunov functions. However, as N gets a little bit higher, the property becomes true for a wider class of functions, referred to as N-step Lyapunov functions (Bobiti and Lazar, 2014). More interestingly, it can be shown (Alamir, 2006; Bobiti and Lazar, 2014) that for stabilizable systems, *any* positive definite function W satisfies the contraction property for appropriate N. Such N's are generally much shorter than the one that would be needed to make standard terminal constraints feasible for a large set of possible initial states.

The difficulty in including the contraction property in the MPC formulation comes from the receding-horizon implementation of the resulting optimal sequence. Indeed, assume that an open-loop *contractive* trajectory is found (at instant k) such that $W(x^{ol}_{k+N}) \leq \gamma W(x^{ol}_k)$, then it might still be true that $\gamma W(x^{ol}_{k+1}) > W(x^{ol}_k)$ since W

is only a finite-step Lyapunov function and hence not monotonically decreasing. This means that if the problem is re-formulated at instant $k+1$ using the constraint $W(x_{k+1+N}) \leq \gamma W(x^{ol}_{k+1})$ then this does not guarantee closed-loop contraction of W. This explains why in the earlier use of the contraction property in MPC formulation (Kothare et al., 2000), two possible alternatives were proposed to enhance closed-loop contraction. In the first, the contractive open-loop trajectory is applied in open-loop until contraction occurs. In the second, the contraction level $\gamma W(x_k = x_{past})$ is memorized and used in formulating the subsequent optimization problems with the constraint $\min_{i=1,\dots,N} W(x_{k+i}) \leq \gamma W(x_{past})$ until contraction occurs at some instant $k+i^*$ at which the updating rule $x_{past} = x_{k+i^*}$ is adopted and the process is repeated. These two alternatives are obviously not satisfactory since in the former, the system is left in open-loop while in the second, the use of memorized level might lead to unfeasibility problem in the presence of disturbance. These drawbacks motivated the contractive scheme proposed in (Alamir, 2007) where no stability-related constraint is used in the MPC formulation.

The present paper improves the formulation proposed in (Alamir, 2007) by using standard cost function together with a stability-dedicated penalty term while in (Alamir, 2007), only the contractive function is used in the cost function which makes the formulation of (Alamir, 2007) exclusively dedicated to stabilization. Moreover, state constraints are considered while (Alamir, 2007) considered only control saturation.

This paper is organized as follows: First, the definitions and notation used throughout the paper are introduced in Section 2 which also introduces the Assumptions needed to derive the main result. Section 3 introduces the proposed contractive MPC formulation together with the main convergence results. Finally Section 4 summarizes the contribution and gives hints for further investigation.

2. DEFINITIONS AND NOTATION

This paper concerns nonlinear systems of the form:

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input. Given a sequence $\boldsymbol{u} := (u^{(1)}, \dots, u^{(N)}) \in \mathbb{R}^{m\times N}$ of future control inputs together with some initial state x, the resulting state trajectory is denoted by $\boldsymbol{x}^{\boldsymbol{u}}(x) := (x^{(1)}, \dots, x^{(N)})$ where $x^{(1)} = f(x, u^{(1)})$ and $x^{(i+1)} = f(x^{(i)}, u^{(i+1)})$. In the sequel, the notation $\boldsymbol{u}_\ell = u^{(\ell)}$ and $\boldsymbol{x}^{\boldsymbol{u}}_\ell(x) = x^{(\ell)}$ is used when needed, namely $\boldsymbol{x}^{\boldsymbol{u}}_\ell(x)$ is the state reached ℓ-steps-ahead starting from the initial state x and applying the sequence of controls $\boldsymbol{u}_1, \dots, \boldsymbol{u}_\ell$.

Regarding the constraints, it is assumed that u belongs to a compact set $\mathbb{U} \subset \mathbb{R}^m$ and that the set of admissible states is given by $\mathbb{G} := \{x \mid g(x) \leq 0\}$. Moreover the following assumption is adopted in the sequel:

Assumption 1. $\mathbb{G}$ is a $\mathbb{U}$-Controlled-Invariant set that contains a neighborhood of the origin.

Note that this Assumption is generally not satisfied if $g(x)$ simply expresses the simple enumeration of physical constraints on x. However, it can be made satisfied by appropriate tightening of the set of constraints as it is shown in the following simple example:

Example 1. Consider the discrete-time version of the system given by $\ddot{r} = u$ where $u \in \mathbb{U} := [-\bar{u}, \bar{u}]$ and $r \in [-\bar{r}, +\bar{r}]$. Take $x = (r, \dot{r})$. If one uses the trivial definition $g(x) = |x_1| - \bar{r}$ then $\mathbb{G}$ does not satisfy Assumption 1. However, if the constraint is tightened so that:

$$g(x) := \begin{pmatrix} |x_1| - \bar{r} \\ x_1 + x_2\tau - Sign(x_2)[\frac{1}{2}\bar{u}\tau^2 + \bar{r}] \end{pmatrix} \tag{2}$$

then the resulting $\mathbb{G}$ meets Assumption 1. This is because the additional constraint limits the speed x_2 so that admissible brake force can avoid the violation of the *original* constraint.

Remark 1. Assumption 1 is a viability assumption that is common i n MPC formulation without terminal constraints and/or cost. See for instance similar statement of this assumption in (Kerrigan and Maciejowski, 2000; Blanchini and Miani, 2008).

Regarding the contraction property, the following assumption is used:

Assumption 2. There exists a positive definite function W, a contraction factor $\gamma \in (0,1)$ and a prediction horizon N such that $\forall x \in \mathbb{G}$, $\exists \boldsymbol{u} \in \mathbb{U}^N$ such that:

$$\forall \ell \in \{1, \dots, N\} \qquad \boldsymbol{x}^{\boldsymbol{u}}_\ell(x) \in \mathbb{G} \tag{3}$$

$$\underline{W}(x, \boldsymbol{u}, N) := \min_{\ell=1}^{N} \Big[W(\boldsymbol{x}^{\boldsymbol{u}}_\ell(x))\Big] \leq \gamma W(x) \tag{4}$$

Remark 2. Note that Assumption 2 implicitly implies Assumption 1. That is the reason why Assumption 1 never appears in the formulation of the results of this contribution. The reason for which Assumption 1 is explicitly stated though is to underline the need for constraints tightening so that the admissible set $\mathbb{G}$ becomes controlled-invariant.

In the sequel, the argument of the minimization problem in (4) over a prediction horizon of length N is denoted by $\ell_{opt}(x, \boldsymbol{u}, N)$. More generally, given a prediction horizon $q \leq N$, the following notation is used:

$$\ell_{opt}(x, \boldsymbol{u}, q) := \underset{\ell \in \{1,\dots,q\}}{\arg\min}\; W(\boldsymbol{x}^{\boldsymbol{u}}_\ell(x)) \tag{5}$$

As mentioned in the introduction, finding a pair (W, N) satisfying Assumption 2 is much easier than looking for standard one-step Lyapunov function as any positive definite function W would be successful candidate for N moderately large. For a specific system, such function can even be linked to some physical considerations. In the absence of such facility, randomized optimization procedure (Alamo et al., 2009) might be used to set the pair (W, N) that satisfies (4) with extremely high probability. Note also that the *machinery* developed in (Grüne et al., 2010) around the concept of multi-step Lyapunov inequality can lead to more checkable conditions that guarantee the contraction property.

It is assumed that a stage cost function $L(x, u)$ is used to express the control objective. For a control sequence $\boldsymbol{u}$, the following notation is used:

$$\Phi(x,\boldsymbol{u},q) := \sum_{\ell=1}^{q} L(\boldsymbol{x}_\ell^{\boldsymbol{u}}(x),\boldsymbol{u}_\ell) \tag{6}$$

Moreover, the following assumption is used regarding the behavior of L inside the admissible domain

Assumption 3. $\exists \bar{L} > 0$ such that:

$$\forall (x,u) \in \mathbb{G}\times\mathbb{U}, \quad 0 \le L(x,u) \le \bar{L} \tag{7}$$

Moreover, $Q(x) := L(x,0)$ is a positive definite function of the state and such that $Q(x) \le L(x,u)$ for all u.

This simply means that L is bounded on the set of admissible pairs (x,u) and contains a positive definite penalty on the state regardless of the control value. In the next section, the proposed MPC formulation is given and the behavior of the resulting closed-loop is analyzed.

3. THE CONTRACTIVE FORMULATION

Let us define for any $z > 0$ and any state $x \in \mathbb{G}$, the following optimization problem, denoted by $\mathcal{P}(x,z)$:

$$\min_{(\boldsymbol{u},q)} \left[J^{(x,z)}(\boldsymbol{u},q)\right] := z\cdot\Phi(x,\boldsymbol{u},q) + \alpha\underline{W}(x,\boldsymbol{u},q) \tag{8}$$

$$\text{under} \qquad \boldsymbol{x}_\ell^{\boldsymbol{u}}(x) \in \mathbb{G} \qquad \forall \ell \in \{1,\dots,q\} \tag{9}$$

$$\text{and} \qquad (\boldsymbol{u},q) \in \mathbb{U}^N \times \{1,\dots,N\} \tag{10}$$

where z is an internal state of the controller with dynamics defined by (15) hereafter. q is the free-prediction horizon which is considered as a decision variable in the proposed formulation. Note that for a given q, only the trajectory over the future interval $[k,k+q]$ is involved in the definition of the cost function (8) and the constraints (9). The two terms involved in the cost function (8) are respectively given by (6) and (4) and represent respectively the stage cost over the prediction horizon of length q and the lowest value of W over the same prediction horizon.

Remark 3. Note that the solution of (8)-(10) involves the integer scalar decision variable q. However, this variable is expected to take only limited number of candidate values. This is the rationale behind the present contribution.

Remark 4. It is assumed hereafter that in the case where several solutions exist to (8)-(10) with different candidate prediction horizons, then the one with the shortest prediction horizon is selected by the optimizer

Let us denote by $\boldsymbol{u}^*(x,z)$ and $q^*(x,z)$ the optimal solutions (if any) of the optimization problem. Moreover, the corresponding value of $\underline{W}$, Φ, J and ℓ_{opt} are denoted by:

$$\underline{W}^*(x,z) := \underline{W}(x,\boldsymbol{u}^*(x,z),q^*(x,z)) \tag{11}$$

$$\Phi^*(x,z) := \Phi(x,\boldsymbol{u}^*(x,z),q^*(x,z)) \tag{12}$$

$$J^*(x,z) := J^{(x,z)}(\boldsymbol{u}^*(x,z),q^*(x,z))) \tag{13}$$

$$\ell^*(x,z) := \ell_{opt}(x,\boldsymbol{u}^*(x,z),q^*(x,z)) \tag{14}$$

The dynamic of the controller's internal state z is given by:

$$z_{k+1} = h(x_k,z_k) := \begin{cases} z_k & \text{if } W(x_k) > z_k \\ \beta z_k & \text{if } W(x_k) \le z_k \end{cases} \tag{15}$$

where $\beta \in (0,1)$ is some fixed constant that can be viewed as a parameter of the controller.

This completely defines the MPC feedback by:

$$z^+ = h(x,z) \tag{16}$$

$$K_{MPC}(x,z) := \boldsymbol{u}_1^*(x,z) \tag{17}$$

In what follows, some preliminary results are derived which are used later in the proof of the main result.

The first result and its Corollary 1 give an explicit and computable upper bound on the optimal cost:

Lemma 1. **If** Assumptions 2, 3 are satisfied **then** $\forall (x,z) \in \mathbb{G}\times\mathbb{R}_+$, $\mathcal{P}(x,z)$ is feasible. Moreover:

$$J^*(x,z) \le zN\bar{L} + \alpha\gamma W(x) \tag{18}$$

and the minimum value of the contractive map W is obtained at the end of the trajectory, namely:

$$\ell^*(x,z) = q^*(x,z) \tag{19}$$

Proof. Feasibility is a direct consequence of Assumption 2 since it guarantees the feasibility of (3) which is the only constraints (9) that may lead to unfeasibility. Moreover, taking any $\boldsymbol{u}$ satisfying the conditions of Assumption 2, the corresponding stage cost is obviously lower than $N\bar{L}$ thanks to (7) of Assumption 3 while the second term $\underline{W}(x,\boldsymbol{u},N)$ is lower than $\gamma W(x)$ by virtue of (4). This proves (18). As for (19), it can be proved by contradiction. Indeed, if $q^*(x,z) > \ell^*(x,z)$, then the candidate solution $(\boldsymbol{u}^c,q^c) := (\boldsymbol{u}^*(x,z),\ell^*(x,z))$ would correspond to a cost function value satisfying

$$J^{(x,z)}(\boldsymbol{u}^c,q^c) \le J^*(x,z) - z\left[\sum_{k=\ell^*(x,z)+1}^{q^*(x,z)} L(\boldsymbol{x}_k^{\boldsymbol{u}^*},0)\right] - \alpha\left[q^*(x,z)-\ell^*(x,z)\right]\underline{W}^*(x,z) < J^*(x,z) \tag{20}$$

which contradicts the optimality of $q^*(x,z)$. □

Corollary 1. Under Assumptions 2 and 3, **if** the following conditions hold

(1) $z < W(x)$ and
(2) $\alpha \ge 2N\bar{L}/(1-\gamma)$

then the optimal solution satisfies the inequality:

$$J^*(x,z) \le \left[\frac{1+\gamma}{2}\right]\alpha W(x) \tag{21}$$

Proof. Since the condition of Lemma 1 are satisfied, inequality (18) holds, namely

$$J^*(x,z) \le zN\bar{L} + \alpha\gamma W(x) \tag{22}$$

and since $z \le W(x)$, one can write:

$$J^*(x,z) \le (N\bar{L} + \alpha\gamma)W(x) \tag{23}$$

and using the assumption $\alpha \ge 2N\bar{L}/(1-\gamma)$, the last inequality becomes:

$$J^*(x,z) \le (\frac{(1-\gamma)\alpha}{2} + \alpha\gamma)W(x) \tag{24}$$

which gives (21) after straightforward manipulation. □

While the two preceding results determine bounds on the optimal cost, the following two lemmas characterize the behavior of two successive values of the optimal cost at instants k where $W(x_k)$ is still greater than z_k. Lemma 2 characterizes this behavior in the case where $q^*(x_k,z_k) > 1$ while Lemma 3 gives this characterization when $q^*(x_k,z_k) = 1$:

Lemma 2. Under Assumptions 2 and 3, **if** the following conditions hold

(1) $q^*(x_k,z_k) > 1$ and

(2) $z_k < W(x_k)$

then the following inequality holds:

$$J^*(x_{k+1}, z_{k+1}) \leq J^*(x_k, z_k) - z_k Q(x_{k+1}) \qquad (25)$$

where $Q(\cdot)$ is the positive definite function invoked in Assumption 3.

Proof. Since $q^*(x_k, z_k) > 1$, an admissible pair $(\boldsymbol{u}^+, q^+)$ for the optimization problem $\mathcal{P}(x_{k+1}, z_{k+1})$ can be given by:

$$q^+ = q^*(x_k, z_k) - 1$$
$$\boldsymbol{u}^+ := \left(\boldsymbol{u}_2^*(x_k, z_k) \ \dots \ \boldsymbol{u}_{q^*(x_k,z_k)}^*(x_k, z_k)\right)$$

But since in this case $z_{k+1} = z_k$ [see (15)] and since the terminal cost is unchanged by virtue of (19) of Lemma 1, the cost of this candidate pair is obviously given by:

$$J^{(x_{k+1},z_{k+1})}(\boldsymbol{u}^+, q^+) = J^*(x_k, z_k) - z_k L(x_{k+1}, \boldsymbol{u}_1^*(x_k, z_k))$$

and using the fact that $L(x, u) \geq L(x, 0) =: Q(x)$ (Assumption 3), the last inequality gives:

$$J^{(x_{k+1},z_{k+1})}(\boldsymbol{u}^+, q^+) = J^*(x_k, z_k) - z_k Q(x_{k+1}) \qquad (26)$$

which obviously leads to (25) by the very definition of optimality. □

Lemma 3. Under Assumptions 2 and 3, **if** the following conditions hold:

(1) $q^*(x_k, z_k) = 1$
(2) $z_{k+1} < W(x_{k+1})$
(3) $\alpha \geq 2N\bar{L}/(1 - \gamma)$

then the following inequality holds:

$$J^*(x_{k+1}, z_{k+1}) \leq J^*(x_k, z_k) - z_k Q(x_{k+1}) \qquad (27)$$

Proof. Since $q^*(x_k, z_k) = 1$ and by virtue of the receding-horizon implementation, one has:

$$J^*(x_k, z_k) = z_k L(x_{k+1}, \boldsymbol{u}_1^*(x_k, u_k)) + \alpha W(x_{k+1}) \qquad (28)$$

therefore, since $L(x, u) \geq Q(x)$, the last inequality implies that:

$$\alpha W(x_{k+1}) \leq J^*(x_k, z_k) - z_k Q(x_{k+1}) \qquad (29)$$

On the other hand, since the conditions of Corollary 1 are satisfied at instant $k + 1$, inequality (21) holds for x_{k+1} and z_{k+1}, namely:

$$J^*(x_{k+1}, z_{k+1}) \leq \left[\frac{1 + \gamma}{2}\right] \alpha W(x_{k+1}) \leq \alpha W(x_{k+1})$$

Combining this last inequality with (29) gives (27). □

Lemma 2 and 3 enables to establish the following corollary:

Corollary 2. Under Assumptions 2 and 3, **If** the a penalty $\alpha \geq 2N\bar{L}/(1-\gamma)$ is used, then for all initial $z_0 > 0$, the set $\{x = 0\}$ is an accumulation set for the closed-loop dynamic system. Namely, there is a subsequence of $\{x_k\}_{k\geq 0}$ that converges to 0.

Proof. Let us adopt the following notation:

$$e_k = W(x_k) - z_k \qquad (30)$$

then the updating rule (15) can be rewritten for clarity using e_k as follows:

$$z_{k+1} = \begin{cases} z_k & \text{if } e_k > 0 \\ \beta z_k & \text{if } e_k \leq 0 \end{cases} \qquad (31)$$

Combining Lemmas 2 and 3 (the conditions of which are satisfied), one can write that:

$$\Big\{e_k > 0 \text{ and } e_{k+1} > 0\Big\} \Rightarrow \qquad (32)$$
$$J^*(x_{k+1}, z_{k+1}) \leq J^*(x_k, z_k) - z_k Q(x_{k+1})$$

Let us denote by $\mathbb{K}_\leq$ the set of instants such that $e_k \leq 0$, more precisely:

$$\mathbb{K}_\leq := \Big\{\kappa_1, \kappa_2, \dots\Big\} \quad \text{where} \quad e_{\kappa_j} \leq 0 \qquad (33)$$

Two situations have to be distinguished:

In the first, the set $\mathbb{K}_\leq$ is finite with cardinality $\sigma = card(\mathbb{K}_\leq)$ while in the second, the set $\mathbb{K}_\leq$ is infinite.

Case where $\mathbb{K}_\leq$ is finite with $\sigma = card(\mathbb{K}_\leq)$

In this case, one has:

$$(\forall k > \kappa_\sigma) \quad e_k > 0 \qquad (34)$$

and therefore, by virtue of (31), it is possible to write:

$$(\forall k > \kappa_\sigma) \quad z_k =: z_\infty := \beta^\sigma z_0 > 0 \qquad (35)$$

and injecting this in (32) enables to write:

$$(\forall k > \kappa_\sigma), \qquad (36)$$
$$J^*(x_{k+1}, z_{k+1}) \leq J^*(x_k, z_k) - z_\infty Q(x_{k+1}) \qquad (37)$$

which obviously proves that the sequence $\{Q(x_k)\}_{k>\kappa_\sigma}$ converges to 0 and so does the sequence $\{x_k\}_{k\geq 0}$

Case where $\mathbb{K}_\leq$ is infinite

In this case, by definition of the updating rule where the second branch is visited an infinite number of times, one obviously has:

$$\lim_{k\to\infty} z_k = 0 \qquad (38)$$

and by definition of the instant e_{κ_j}, it comes that:

$$\lim_{j\to\infty} W(x_{\kappa_j}) \leq \lim_{j\to\infty} z_{\kappa_j} = 0 \qquad (39)$$

which shows that there is a partial sequence of $\{x_k\}_{k\geq 0}$ that converges to 0. □

Corollary 2 proves that the trajectory of the state visits regularly an always smaller neighborhood of the targeted state $x = 0$ at an increasing infinite instants of time. It remains to analyze the asymptotic behavior of excursion of the state trajectory between these instants. In order to do this, a local assumption is needed regarding the property of the system in an arbitrary small neighborhood of the origin:

Assumption 4. The origin is locally N-step stabilizable. More precisely, there is a local neighborhood $\mathcal{V}$ of the origin such that for all $x_0 \in \mathcal{V}$, there is an admissible control sequence $\boldsymbol{u}$ such that $\boldsymbol{\Pi}_c\,[x_N^{\boldsymbol{u}}(x_0)] = 0$ [where $\Pi_c(x)$ is the controllable substate corresponding to x of the linearized system around the origin]. Moreover, the corresponding state trajectory is entirely contained in $\mathbb{G}$.

Remark 5. It is worth underlying that this assumption is much less stringent than the N-reachability assumption (in the large) as used in the early formulations of stable NMPC (Keerthi and Gilbert, 1988; Mayne and Michalska, 1990). Indeed, the assumption used here is only local and imposes the reachability-in-N-step assumption only on a small neighborhood of the targeted state that can be as small as necessary.

We now have all we need to state the main result of this contribution:

Proposition 1. Assume that:

(1) Assumptions 2-4 are satisfied.
(2) The penalty α involved in the cost function (8) is such that
$$\alpha \geq \frac{2N\bar{L}}{1-\gamma} \tag{40}$$

Then $x = 0$ is asymptotically stable for the closed-loop associated to the MPC law defined by (8)-(10) for all initial states (x, z) such that $x \in \mathbb{G}$ and $z > 0$.

Proof. We shall characterize the behavior of the state trajectory over time. To do this, let us divide the time instants into two sets, namely:
$$\mathbb{K}_{\leq} := \Big\{k \in \mathbb{N} \mid e_k := W(x_k) - z_k \leq 0\Big\} \tag{41}$$
$$\mathbb{K}_{>} := \Big\{k \in \mathbb{N} \mid e_k := W(x_k) - z_k > 0\Big\} \tag{42}$$
Note that we already encountered $\mathbb{K}_{\leq}$ in the proof of Corollary 2. The behavior of the state trajectory over the set $\mathbb{K}_{\leq}$ is easy to characterize since by virtue of (35), one has:
$$(\forall k \in \mathbb{K}_{\leq}) \qquad W(x_k) \leq \left[\beta^{m_k - 1}\right] z_0 \tag{43}$$
where for all $k \in \mathbb{K}_{\leq}$, the integer m_k denotes the order of k in $\mathbb{K}_{\leq}$.

As for $\mathbb{K}_{>}$, the two cases regarding whether $\mathbb{K}_{\leq}$ is finite or not have to be distinguished.

Case where $\mathbb{K}_{\leq}$ is finite with cardinality σ.

In this case, the proof of Corollary 2 already showed that the inequality (37) becomes satisfied after a finite number of steps $k = \kappa_\sigma + 1$ and therefore, the behavior of the closed-loop trajectory is such that:
$$\sum_{k=\kappa_\sigma+2}^{\infty} Q(x_k) \leq \frac{1}{z_\infty}\left[J^*(x_{\kappa_\sigma+1}, z_\infty)\right] < \infty \tag{44}$$
where $z_\infty := \beta^\sigma z_0 > 0$.

Case where $\mathbb{K}_{\leq}$ is infinite.

Note that in this case, thanks to (39) we have the characterization of the behavior over $\mathbb{K}_{\leq}$. As for the behavior over instants in $\mathbb{K}_{>}$, insight can be obtained by observing that between any two successive instants $\kappa_j, \kappa_{j+1} \in \mathbb{K}_{\leq}$, a constant and non vanishing $z_{\kappa_j+1} = C_j \neq 0$ applies and therefore, one obtains the same optimal solution (and hence the same state trajectory) if the cost function is divided by C_j to get the modified cost
$$\Phi(x, \boldsymbol{u}, q) + \left[\frac{\alpha}{C_j}\right] \underline{W}(x, \boldsymbol{u}, q) \tag{45}$$
Moreover, since we know that $\ell^* = q^*$, the second term can simply be replaced by a penalty on the final value to get the following modified cost function:
$$\Phi(x, \boldsymbol{u}, q) + \left[\frac{\alpha}{C_j}\right] W(x, \boldsymbol{u}, q) \tag{46}$$
Now since $\lim_{j\to\infty} C_j = 0$, the corresponding MPC formulation behaves asymptotically as a MPC formulation with final equality constraints on the controllable sub-state. We know that such formulation under Assumption 4 (that becomes true for sufficiently high j for which $x_{\kappa_j} \in \mathcal{V}$) and the positive definiteness of L used in Φ leads to well qualified and stable behavior over the interval $[\kappa_j+1, \kappa_{j+1}]$ (Mayne et al., 2000). This clearly ends the proof. □

Note that the condition (40) of Proposition 1 is a quantified realization of the stabilizing role of terminal penalty in the context of absence of terminal constraint as suggested by Grüne et al. (2010) (See the discussion of Section 8.2 regarding this issue).

4. CONCLUSION AND FUTURE WORK

In this paper, a new contraction-based NMPC formulation is proposed. The formulation uses the contraction property in the proof of the convergence of the closed-loop system but does not add any stability-related terminal constraint that involves the contraction property. The assumptions needed for the success of the formulation are rather standard and should be satisfied for short prediction horizons making the formulation adapted to situations where fast computation are necessary.

Nevertheless, although the expected successful prediction horizon should be small, a computational issue remains to be handled since the optimization problem involves the integer variable q. Investigation regarding the ways to efficiently tackle this issue without the cumbersome machinery of mixed continuous-integer programming is under investigation. Mainly, a two-step procedure leading to solving two successive (but potentially different) **fixed**-prediction horizon should be possible. The corresponding algorithm together with an illustrative examples should be soon available.

REFERENCES

Alamir, M. (2006). *Stabilization of Nonlinear Systems Using Receding-Horizon Control Schemes: A Parametrized Approach for Fast Systems*. Springer. Lecture Notes in Control and Identification Sciences, number 339. ISBN : 1-84628-470-8.

Alamir, M. (2007). A low dimensional contractive nmpc scheme for nonlinear systems stabilization: Theoretical framework and numerical investigation on relatively fast systems. In *Assessment and Future Directions of Nonlinear Model Predictive Control*, 523–535. Springer.

Alamir, M. and Bornard, G. (1995). Stability of a truncated infinite constrained receding horizon scheme: the general discrete nonlinear case. *Automatica*, 31(9), 1353 – 1356.

Alamo, T., Tempo, R., and Camacho, E. (2009). Randomized strategies for probabilistic solutions of uncertain feasibility and optimization problems. *Automatic Control, IEEE Transactions on*, 54(11), 2545–2559.

Blanchini, F. and Miani, S. (2008). *Set-theoretic methods in control. Systems & control: foundations & applications*. Birkhäuser Boston, MA.

Bobiti, R. and Lazar, M. (2014). On the computation of lyapunov functions for discrete-time nonlinear systems. In *System Theory, Control and Computing (ICSTCC), 2014 18th International Conference*, 93–98.

Boccia, A., Grüne, L., and Worthmann, K. (2014). Stability and feasibility of state constrained {MPC} without stabilizing terminal constraints. *Systems & Control Letters*, 72, 14 – 21.

Grimm, G., Messina, M.J., Tuna, S.E., and Teel, A.R. (2005). Model predictive control: for want of a local control lyapunov function, all is not lost. *IEEE Transactions on Automatic Control*, 50(5), 546–558.

Grüne, L. and Pannek, J. (2011). *Nonlinear Model Predictive Control. Theory and Algorithms.* Springer-Verlag.

Grüne, L., Pannek, J., Seehafer, M., and Worthmann, K. (2010). Analysis of unconstrained nonlinear mpc schemes with time-varying control horizon. *SIAM Journal on Control and Optimization*, 48(8), 4938–4962.

Jadbabaie, A. and Hauser, J. (2005). On the stability of receding horizon control with a general terminal cost. *IEEE Transactions on Automatic Control*, 50(5), 674–678.

Keerthi, S.S. and Gilbert, E.G. (1988). Optimal infinite horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving horizon approximations. *Journal of Optimization Theory and Applications*, 57, 265–293.

Kerrigan, E.C. and Maciejowski, J.M. (2000). Invariant sets for constrained nonlinear discrete-time systems with application to feasibility in model predictive control. In *Decision and Control, 2000. Proceedings of the 39th IEEE Conference on*, volume 5, 4951–4956.

Kothare, S., de Oliveira, L., and Morari, M. (2000). Contractive model predictive control for constrained nonlinear systems. *IEEE Transations on Automatic Control*, 45.

Lazar, M. and Spinu, V. (2015). Finite-step terminal ingredients for stabilizing model predictive control. *IFAC-PapersOnLine*, 48(23), 9 – 15. 5th IFAC Conference on Nonlinear Model Predictive Control NMPC 2015 Seville, Spain, 1720 September 2015.

Mayne, D.Q. and Michalska, H. (1990). Receding horizon control of nonlinear systems. *IEEE Transactions on Automatic Control*, 35, 814–824.

Mayne, D.Q., Rawlings, J., Rao, C.V., and Scokaert, P.O.M. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36, 789–814.