From Certification of Algorithms To Certified MPC: The Missing Links ⋆

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Abstract: Deriving certification bounds for optimization algorithms is an active research area in the control community. This is mainly impulsed by the use of on-line optimization algorithms in real-time MPC through limited computation power. However, the way such bounds are then used to derive a convergence certification for MPC frameworks is still not sufficiently mature. This paper contributes in clarifying what are the unavoidable additional ingredients that need to be combined with any algorithm's certification bound in order to derive a relevant certification result for the MPC-based closed-loop performance. Moreover, the paper gives such a general certification result based on these ingredients for any pair of certified algorithm and provably stable ideal MPC formulation. The proposed framework is then instantiated to the particular case of linear MPC and a simple example is given to illustrate the introduced concepts.

Keywords: MPC Certification, State-Dependent Updating Period, Real-Time Implementation.

1. INTRODUCTION

The success of Model Predictive Control design (Mayne et al. (2000)) expanded its domain of application to areas where high sampling rates are necessary. Since MPC design is based on the repetitive solution of optimization problems, the need for high sampling rates requires the optimization algorithms to be interrupted. This raises the problem of certification which can be stated in at least three different ways:

(a) Given a standard constant control updating period scheme and a given computation power, what result can be guaranteed regarding the closed-loop behavior of the MPC-controlled system?

(b) For a given computation power, what is the *control updating strategy* to adopt in order to achieve the best possible certification?

(c) Given a desired certification level what is the computation power and the control updating strategy that make possible the achievement of this level?

One of the key properties that partially determine the answers to the above questions is the certification bounds associated to the optimization algorithm being used. These bounds give the minimum number of iterations (of that specific algorithm) that would guarantee the achievement of some desired precision on the value of the cost function and the constraints satisfaction. This issue has received increasing attention in the past few years and is still an active research area (Jones et al. (2012); Richter et al. (2012); Giselsson (2012); Bomporad and Patrinos (2012)).

Although of great importance, the algorithm's certification bound is only a single ingredient in the statement of any possible answer to any of the questions raised above. This paper shows that *all* of the following items must be present in any possible *MPC certification* statement:

(1) *The ideal MPC Stability*-related quantities,
(2) *The computation power*,
(3) *The uncertainty level of the future prediction*,
(4) *The quality of the initial guess*,
(5) *The algorithm's certification bounds*.

In other words, the aim of this paper is to convince the reader that:

> Any MPC certification statement in which one of the above ingredients is absent is at least *incomplete* if not *wrong*.

This paper gives a concrete instantiation regarding the way these ingredients interact in the MPC certification statement. The main result clearly shows that a state-dependent control updating period can be derived that can be viewed as the theoretical foundation underlying the heuristics proposed by the author in some recent contributions (Alamir (2008, 2013)) in order to dynamically adapt the control updating period based on the on-line measured behavior of the cost function.

It is worth noting that the framework proposed in this paper applies regardless of the optimization algorithm being used in the MPC implementation provided that a certification bound can be derived to be used in the formulation of the main result.

This paper is organized as follows: First some definitions and notation are stated in section 2. The working assumptions that are needed to establish the main results

⋆ The author is grateful to Prof. M. Morari for the fruitful discussion regarding this topic during the ECC2014 conference. This discussion was the starting point for the investigation proposed in this paper.

are given in section 3. Section 4 gives the main MPC certification result. Section 5 gives a complete instantiation in the linear case while section 6 concludes the paper and give some ideas for further investigation.

2. DEFINITIONS AND NOTATION

We consider a discrete time dynamic system given by:

$$z_{k+1} = f(z_k, u_k, w_k) \tag{1}$$

where $z_k \in \mathbb{R}^{n_z}$ is the state vector at instant $k\tau_s$, $u_k \in \mathbb{R}^{n_u}$ is the control input applied over $[k\tau_s, (k+1)\tau_s]$ while $w_k \in \mathbb{R}^{n_w}$ denotes the disturbance vector. The period $\tau_s > 0$ is the sampling period that defines the piecewise constant control. This basic sampling period is to be distinguished from the control updating period that is invoked later in the sequel [see equation (30)].

Given a prediction horizon's length N_p, the boldfaced notation $\mathbf{u}_k$ refers to a sequence of N_p successive inputs, namely:

$$\mathbf{u}_k = \{u_k, u_{k+1}, \dots, u_{k+N_p-1}\} \in [\mathbb{R}^{n_u}]^{N_p} \tag{2}$$

while the sequence of the i-th first inputs is denoted by:

$$\mathbf{u}_k^{(1\to i)} := \{u_k, \dots, u_{k+i-1}\} \tag{3}$$

Note that the design of MPC for disturbed dynamic systems, or for system having dynamically varying set-point needs a prediction model for the corresponding unknown vector. This model is denoted by:

$$w_{k+1} = f_d(w_k) \tag{4}$$

Remark 1. Note that as far as the MPC dsign is concerned, the set point can be viewed as a component of the disturbance vector as it represents a measured but unpredictable signal that affects the value of the predicted cost function in an MPC design.

Using (1) and (4) together with the extended state:

$$x := \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{R}^n \quad ; \quad n = n_z + n_w \tag{5}$$

the dynamic system used by the MPC solver can be given in the following compact form:

$$x_{k+1} = F(x_k, u_k) \tag{6}$$

where x gathers now the physical state vector, the signal needed to define the desired behavior (including future behavior of set-points) and the disturbance vector. From this, it comes that a general formulation of large class of MPC control objectives (stabilization, tracking, economic) can be formulated using a cost function that takes at instant $k\tau_s$ the following form:

$$J(\mathbf{u}, x_k) := \sum_{i=1}^{N_p} \ell(\hat{x}^{\mathbf{u}}_{k+i|k}, u_{k+i-1}) \tag{7}$$

where $\hat{x}^{\mathbf{u}}_{k+i|k}$ denotes the prediction of the state at instant $(k+i)\tau_s$ when starting at instant $k\tau_s$ at the state x_k and under the control sequence $\mathbf{u}$. The cost function (7) is to be minimized while meeting the constraints defined by:

$$(\forall i \in I_h \cup I_s) \qquad c_i(\mathbf{u}, x_k) \le 0 \tag{8}$$

where I_h [resp. I_s] is the set of indices of the hard [resp. soft] constraints so that $I_h \cup I_s$ defines a partition of the set of constraints indices $\{1, \dots, n_c\}$. In the sequel, the optimization problem defined by (7)-(8) is denoted by $\mathcal{P}(x_k)$.

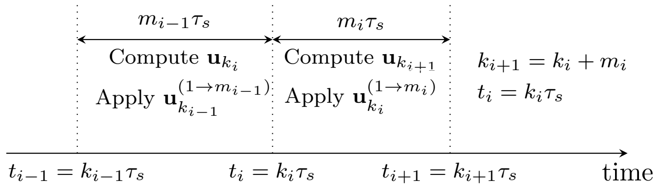


Fig. 1. The real-time implementation scheme: Definition of the control updating instants $t_i = k_i\tau_s$.

The real-time implementation of the MPC controller is depicted in Figure 1, namely:

✓ Computation of new optimal solutions takes place on successive time intervals of variable durations $m_i\tau_s$. These intervals are denoted by $[t_i, t_{i+1}]$ where $t_i = k_i\tau_s$ and $k_{i+1} = k_i + m_i$ (see Figure 1).

✓ During each interval $[t_i, t_{i+1}]$, the first m_i inputs $\mathbf{u}_{k_i}^{(1\to m_i)}$ of the sequence $\mathbf{u}_{k_i}$ (precomputed during the past interval $[t_{i-1}, t_i]$ based on the predicted state $\hat{x}(t_i)$] are applied while iterations are performed in order to compute the new sequence $\mathbf{u}_{k_{i+1}}$ based on the predicted value $\hat{x}(t_{i+1})$ of the state at instant $t_{i+1} := k_{i+1}\tau_s$, namely:

$$\hat{x}(t_{i+1}) := x^{\mathbf{u}_{k_i}}_{k_{i+1}|k_i} \tag{9}$$

✓ The m_{i+1} first inputs $\mathbf{u}_{k_{i+1}}^{(1\to m_{i+1})}$ of the so computed sequence $\mathbf{u}_{k_{i+1}}$ are applied during the next updating period $[t_{i+1}, t_{i+2}]$ and so on.

The choice of the integers m_i that define how long the computation can last before the control law is updated, whether it can be state dependent (leading to $m_i(x_{k_i})$) and whether such m_i's can be found such that convergence can be certified is in the heart of this paper's contribution.

3. WORKING ASSUMPTIONS

It is assumed that a specific algorithm (fast gradient, SQP, interior point, etc.), denoted hereafter by $\mathcal{A}$ is used to solve the optimization problem (7)-(8) defined by the cost function $J(\cdot, \hat{x}(t_i))$ and the constraints $c(\cdot, \hat{x}(t_i))$. It is also assumed that one disposes of certification bounds for $\mathcal{A}$ in the following sense:

Assumption 3.1. (Algorithm's certification bound).
For any $\epsilon_0 > 0$ and $\epsilon_c > 0$, any extended state x_k and any initial guess $\mathbf{u}^{(0)}$, there is a computable number of iterations $N(\mathbf{u}^{(0)}, x_k, \epsilon_0, \epsilon_c)$ beyond which, the algorithm $\mathcal{A}$ used to solve (7)-(8) reaches an iterate $\mathbf{u}^{(i)}$ satisfying for all $i \ge N(\mathbf{u}^{(0)}, x_k, \epsilon_0, \epsilon_c)$:

$$|J(\mathbf{u}^{(i)}, x_k) - J(\mathbf{u}^{opt}(x_k), x_k)| \le \epsilon_0 \tag{10}$$

$$(\forall i \in I_s) \qquad c_i(\mathbf{u}^{(i)}, x_k) \le \epsilon_c \tag{11}$$

$$(\forall i \in I_h) \qquad c_i(\mathbf{u}^{(i)}, x_k) \le 0 \tag{12}$$

where $\mathbf{u}^{opt}(x_k)$ denotes the optimal solution of the problem $\mathcal{P}(x_k)$. Moreover, there is a specific generation of the initial guess, denoted hereafter by $\mathbf{u}_k^{init}$ such that for any

x_k *of interest*, there is a computable upper bound $N(\epsilon_0, \epsilon_c)$ that depends only on the quality indicators ϵ_0 and ϵ_c. ♢

Remark 2. Among many other possibilities, the specific generation of $\mathbf{u}_k^{init}$ may be state independent (for instance using systematically the cold start $\mathbf{u}_k^{init} = 0$) or it can be linked to the previous iterates (hot start) or the unconstrained solution to the optimization problems. This choice impacts the possibility to compute the resulting $N(\epsilon_0, \epsilon_c)$.

Remark 3. Note that the use of the state and control independent certification bound $N(\epsilon_0, \epsilon_c)$ implicitly assumes that all the states that will be visited belong to some compact set $\mathbb{X}_{max} \subset \mathbb{R}^n$ while the visited control inputs are limited by a part of the set of hard constraints. This can be true only for a compact set $\mathbb{X}_0 \subset \mathbb{X}_{max}$ of initial conditions. The mechanism of the proof is then rather standard: It is first proved that if the state starts in $\mathbb{X}_0$, then the certification bounds computed for $\mathbb{X}_{max}$ can be used to prove that the cost function decreases. Therefore, if the sets $\mathbb{X}_0$ and $\mathbb{X}_{max}$ are defined as level sets of the cost function, this would prove that the next state is still in $\mathbb{X}_{max}$ and the argument can be repeated. A complete handling of this feature is conducted in (Alamir (2015)) while the technicalities are avoided here to concentrate on the main streamline.

Definition 3.1. The algorithm $\mathcal{A}$ is said to be asymptotic if infinite precision on the cost function needs infinite number of iterations, namely $\lim_{\epsilon_0 \to 0} N(\epsilon_0, \epsilon_c) = \infty$. ♢

This definition holds obviously for the fast-gradient based algorithms.

Assumption 3.2. (The available computation power) The system is controlled with a computation facility that performs a single iteration of $\mathcal{A}$ in τ_c time units.

Assumption 3.3. (Uncertainty-induced prediction error). There is $E_1 \geq 0$ such that the prediction error satisfies:

$$\|\hat{x}^{\mathbf{u}}_{k+j|k} - x_{k+j}\| \leq E_1 \times j \tag{13}$$

for any x_k and any $\mathbf{u}$ of interest.

Note that E_1 depends on the basic sampling period as well as on the possible amplitude of set-point increment and/or the level of uncertainties.

Assumption 3.4. (Stability of ideal MPC formulation). It is assumed that there is a nonnegative function $q(\cdot)$ defined on $\mathbb{R}^n$ such that the ideal solution to (7)-(8) satisfies:

$$\begin{aligned} &J(\mathbf{u}^{opt}(t_{i+1}), \hat{x}(t_{i+1})) - J(\mathbf{u}^{opt}(t_i), x(t_i)) \leq \\ &\leq -\Delta(m_i, x(t_i)) := -\sum_{j=1}^{m_i} q(\hat{x}^{opt}_{k_i+j-1|k_i}) \end{aligned} \tag{14}$$

where $\hat{x}^{opt}_{k_i+j|k_i}, j = 1, \dots, m_i$ is the predicted state trajectory over the interval $[t_i, t_{i+1}]$ starting from $x(t_i) = x_{k_i}$ under the optimal control $\mathbf{u}^{opt}_{k_i}$ while $\hat{x}(t_{i+1}) := \hat{x}^{opt}_{k_i+m_i|k_i}$.

Remark 4. Note that this last assumption states the decreasing property of the ideal MPC open-loop trajectory. In the seminal survey paper (Mayne et al. (2000)) where the standard case $m_i = 1$ for all i is used, the term $\Delta(1, x)$ is simply given by $q(x) := \ell(x, \kappa_N(x))$ where $\kappa_N(x)$ is the optimal ideal MPC control. Note that for this to be possible, appropriate final constraints are needed that are supposed to be part of the n_c constraints (8) following the recommendations of (Mayne et al. (2000)). ♢

Remark 5. Note that the inequality (14) involves only the model used by the MPC as the predicted future states $\hat{x}^{opt}_{k_i+j|k_i}$ (and in particular $\hat{x}(t_{i+1})$) denote model-based predicted states and not the real future state for which the inequality (14) would not be necessarily valid. ♢

Assumption 3.5. (Bounded steering rate).
There is $D > 0$ such that for any pair $(\mathbf{u}, x_k)$ satisfying the hard constraints, one has for all $j \in \{1, \dots, N_p\}$:

$$q(x^{\mathbf{u}}_{k+j|k}) \geq \max\{0, q(x_k) - D \times j\} \tag{15}$$

where $q(\cdot)$ is the map invoked in Assumption 3.4. ♢

This last assumption states that if $q(x_k)$ is high, one needs high number of iterations j ($\geq q(x_k)/D$) before reaching values of $x^{\mathbf{u}}_{k+j|k}$ where q vanishes.

Assumption 3.6. There are two scalar positive definite maps K_0 and K_c such that for any pair $(\mathbf{u}, x^{(1)})$ and $(\mathbf{u}, x^{(2)})$ of interest, the following inequalities hold:

$$|J(\mathbf{u}, x^{(1)}) - J(\mathbf{u}, x^{(2)})| \leq K_0(\|x^{(1)} - x^{(2)}\|) \tag{16}$$

$$\|c(\mathbf{u}, x^{(1)}) - c(\mathbf{u}, x^{(2)})\|_\infty \leq K_c(\|x^{(1)} - x^{(2)}\|) \tag{17}$$

Note that any of the above assumptions can hardly be absent from any realistic context for which MPC certification can reasonably be expected. That is the reason why the results of the next section do have a quite general scope.

4. MAIN RESULTS

Note that the complete definition of the MPC implementation described in section 2 depends on how the successive integers m_i are defined. Recall that $m_i\tau_s$ defines the length of the interval $[t_i, t_{i+1}]$ during which the computation of sub-optimal solutions to the problems $\mathcal{P}(\hat{x}(t_{i+1}))$ is performed. It results that one reasonable parametrization of the integers m_i can be obtained by taking m_i to be the minimum value that achieves a prescribed precision pair $(\epsilon_0^{(i+1)}, \epsilon_c)$ on the solution to the optimization problem $\mathcal{P}(\hat{x}(t_{i+1}))$, namely:

$$m_i := \bar{m}(\epsilon_0^{(i+1)}, \tau_c) := \lfloor \frac{\tau_c}{\tau_s} N(\epsilon_0^{(i+1)}, \epsilon_c) \rfloor \tag{18}$$

since such m_i corresponds to a computation time $m_i\tau_s$ that is greater than the necessary time to perform $N(\epsilon_0^{(i+1)}, \epsilon_c)$ iterations on a computation facility that needs τ_c time units to perform a single iteration (Assumption 3.2).

Remark 6. Note that in the parametrization (18), only $\epsilon_0^{(i+1)}$ depends on i. This is because the admissible tolerance on the cost function values is state-dependent (far from the objective, even rough sub-optimal solutions may decrease the cost) while the tolerance on the soft constraints satisfaction can be viewed as part of the specification. ♢

The iterate that one gets after these iterations is denoted hereafter by:

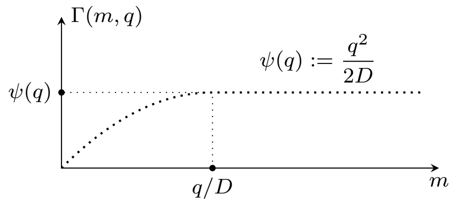


Fig. 2. Illustration of Lemma 4.1.

$$\mathbf{u}^*_{k_{i+1}} := \mathcal{A}\left(\mathbf{u}^{init}_{k_{i+1}}, N(\epsilon_0^{(i+1)}, \epsilon_c)\right) \tag{19}$$

where the r.h.s notation refers to N successive iterations of the algorithm $\mathcal{A}$ starting with the initial guess $\mathbf{u}^{init}_{k_{i+1}}$ (see remark 2). Note that $\mathbf{u}^*_{k_{i+1}}$ is an approximation of the optimal solution $\mathbf{u}^{opt}_{k_{i+1}}$ to the problem $\mathcal{P}(\hat{x}(t_{i+1}))$ and that the quality of this approximation depends on the pair $(\epsilon_0^{(i+1)}, \epsilon_c)$.

Lemma 4.1. If Assumption 3.4 is satisfied, then for any i, the term $\Delta(\cdot)$ involved in (14) satisfies (Figure 2):

$$\Delta(m_i, x(t_i)) \geq \Gamma\left(m_i, q(x(t_i))\right) \tag{20}$$

where:

$$\Gamma(m,q) := \begin{cases} m(q - \frac{D}{2}m) & \text{if } m \leq \frac{q}{D} \\ \psi(q) := \frac{q^2}{2D} & \text{otherwise} \end{cases} \tag{21}$$

Proof. Use the inequality (15) in the definition (14). The max operator involved in (15) leads to the conditional definition (21). □

In the sequel, the short notation $q(t_i) = q(x(t_i))$ is used.

Lemma 4.2. If Assumptions 3.1, 3.3, 3.4 and 3.6 then the following inequality

$$J(\mathbf{u}^*_{k_{i+1}}, x(t_{i+1})) - J(\mathbf{u}^*_{k_i}, x(t_i)) \leq \epsilon_0^{(i)} + R(\epsilon_0^{(i+1)}, q(t_i), \tau_c) \tag{22}$$

in which:

$$R(\epsilon_0, q, \tau_c) := \epsilon_0 + K_0(E_1 \bar{m}(\epsilon_0, \tau_c)) - \Gamma(\bar{m}(\epsilon_0, \tau_c), q) \tag{23}$$

holds for any sequence $\{\epsilon_0^{(i)}\}_{i\geq 1}$ of assigned precisions.

Proof. Applying successively Assumption 3.6 and 3.3:

$$J(\mathbf{u}^*_{k_{i+1}}, x(t_{i+1}) \leq J(\mathbf{u}^*_{k_{i+1}}, \hat{x}(t_{i+1})) + K_0(E_1 \bar{m}(\epsilon_0^{(i+1)}, \tau_c)) \tag{24}$$

and thanks to (10) of Assumption 3.1, one has:

$$J(\mathbf{u}^*_{k_{i+1}}, \hat{x}(t_{i+1})) \leq J(\mathbf{u}^{opt}(t_{i+1}), \hat{x}(t_{i+1})) + \epsilon_0^{(i+1)} \tag{25}$$

This together with (14) of Assumption 3.4 leads to:

$$\begin{aligned} & J(\mathbf{u}^*_{k_{i+1}}, \hat{x}(t_{i+1})) \leq \\ & J(\mathbf{u}^{opt}(t_i), x(t_i)) - \Delta(m_i, x(t_i)) + \epsilon_0^{(i+1)} \\ & \leq J(\mathbf{u}^*_{k_i}, x(t_i)) + \epsilon_0(i) - \Delta(m_i, x(t_i)) + \epsilon_0^{(i+1)} \end{aligned} \tag{26}$$

where the last inequality is obtained by applying (10) of Assumption 3.1 to $\mathcal{P}(x(t_i))$. Gathering (24)-(26) together with (20) and (18) achieves the proof. □

Note that the inequality (22) is fundamental in the study of the closed-loop behavior as it shows the contributions of all the ingredients of the problem to the dynamic of the cost function values between two successive control updating instants, namely:

- ✓ The Ideal MPC property (through $q(x)$)
- ✓ The computation power (through τ_c)
- ✓ The level of future prediction error (through E_1)
- ✓ The current precision $\epsilon_0^{(i)}$
- ✓ The certification bound (through $\bar{m}(\epsilon_0, \tau_c)$)

More precisely, we have the following result:

Proposition 4.1. Assume that the working Assumptions stated in section 3 are satisfied. Given a computation device characterized by τ_c (see Assumption 3.2) and a pre-defined soft constraints satisfaction level $\epsilon_c > 0$, **If** the following conditions hold:

(1) there exist $q_{min} > 0$ and $\gamma_1, \gamma_2 \in (0,1)$ such that for all $q \geq q_{min}$, the inequality

$$R(\epsilon_0, q, \tau_c) \leq -\gamma_1 \psi(q) \tag{27}$$

admits a solution:

$$\epsilon_0^*(q) \in [0, \gamma_1\gamma_2\psi(q)] \tag{28}$$

(2) The initial guess is such that

$$\epsilon_0^{(0)} \leq \gamma_1\gamma_2\psi(q(x(t_0))) \tag{29}$$

then the MPC based on the state-dependent updating period given by:

$$\tau_i := m_i\tau_s \quad ; \quad m_i := \bar{m}\left(\epsilon_0^{(i+1)}, \tau_c\right) \tag{30}$$

with

$$\epsilon_0^{(i+1)} := \epsilon_0^*(q(x(t_i))) \tag{31}$$

stabilizes the set $\mathbb{X}$ given by:

$$\mathbb{X} := \{x \in \mathbb{R}^n \mid \quad q(x) \leq q_{min}\} \tag{32}$$

and leads to closed-loop trajectory satisfying all the hard constraints while satisfying the inequality

$$\max_{i \in I_h} \left[c_i(\mathbf{u}^*_{k_i}, x(t_i))\right] \leq \epsilon_c + K_c(E_1\tau_i) \tag{33}$$

on the soft constraints fulfillment. ♢

Proof. Note first of all that thanks to (29) and the fact that all the future $\epsilon_0^{(i+1)}$ are defined by (31) in which ϵ_0^* satisfies (28), one has for all $i \geq 0$ that:

$$\epsilon_0^{(i)} \leq \gamma_1\gamma_2\psi(q(x(t_i)))$$

Using this inequality in (22) together with (27) enables to state that as long as $q(t_i) \geq q_{min}$, on has

$$J(\mathbf{u}^*_{k_{i+1}}, x(t_{i+1})) - J(\mathbf{u}^*_{k_i}, x(t_i)) \leq -\gamma_1(1-\gamma_2)\psi(q(x(t_i)))$$

and since ψ is a positive definite function of q (see Figure 2) and the fact that the cost function is bounded below (well posedness), this clearly shows the first part of the proposition. Regarding the constraints, note that the hard constraints are satisfied by definition of the algorithm's certification bound [inequality (12)] while the soft constraints satisfy by the same definition inequality (11) which, combined to (17) completes the proof. □

In the following section, the conditions are given under which q_{min} can be found that satisfies the requirement of the Proposition and the procedure by which a state-dependent required precisions $\epsilon_0^*(q(x))$ can be computed off-line leading to a state-dependent control updating period is explicitly given.

4.1 Discussion

In order to get a deeper understanding of Proposition 4.1, let us examine the main inequality (27) that is reproduced here with the detailed definition (23) of the $R(\cdot)$ term:

$$\epsilon_0 + K_0(E_1\bar{m}(\epsilon_0,\tau_c)) \leq \Gamma(\bar{m}(\epsilon_0,\tau_c),q) - \gamma_1\psi(q) \quad (34)$$

This inequality has to be satisfied for all $q \geq q_{min}$ and a corresponding appropriate ε_0. In particular, it must hold for q_{min}, namely:

$$\epsilon_0 + K_0(E_1\bar{m}(\epsilon_0,\tau_c)) \leq \Gamma(\bar{m}(\epsilon_0,\tau_c),q_{min}) - \gamma_1\psi(q_{min}) \quad (35)$$

The terms involved in the inequality (34) are shown in Figure 3, more precisely:

- **Figure 3.(a)** shows how the l.h.s of (34) depends on the precision ϵ_0. Note that this dependence does not involve the value of q and only involves the computational power through τ_c, the predefined precision level ϵ_c on the satisfaction of the soft constraints and the uncertainty level E_1. The certification bound appears through the term $\bar{m}(\epsilon_0,\tau_c)$ and is represented by a decreasing function as the number of iterations decreases when ϵ_0 increases. Note that even when ϵ_0 goes to infinity, there is still a minimum number of iterations that would be needed to handle the soft constraints that are not representable through simple set on which projection can be done almost without cost. Figure 3.(a) shows that the l.h.s of (34) can not be steered lower than $\check{\ell}_0$ that corresponds to precision $\check{\epsilon}_0$.

- Consequently, the r.h.s of (34) and (35) must necessarily be greater than $\check{\ell}_0$, namely:

$$\check{\ell}_0 \leq \Gamma(\bar{m}(\epsilon_0,\tau_c),q_{min}) - \gamma_1\psi(q_{min}) \quad (36)$$

and since $\Gamma(\bar{m}(\epsilon_0,\tau_c),q_{min}) \leq \psi(q_{min})$ (see Figure 2), the last inequality implies that q_{min} is bounded below according to:

$$\psi(q_{min}) \geq \frac{\check{\ell}_0}{1-\gamma_1} \quad (37)$$

This is summarized in the following statement:

Proposition 4.2. Given the computation power parameter τ_c, the level of uncertainty E_1, a given tolerance on the soft constraints satisfaction ϵ_c, any q_{min} satisfying the condition of Proposition 4.1 is bounded below by (37) where $\check{\ell}_0$ is given by

$$\check{\ell}_0 := \min_{\epsilon_0\geq 0}\left[\epsilon_0 + K_0(E_1\bar{m}(\epsilon_0,\tau_c))\right] \quad (38)$$

Proposition 4.2 indicates that given $(\tau_c, E_1, \epsilon_c)$ the terminal set $\mathbb{X}$ invoked in proposition 4.1 is necessarily wider that

$$\check{\mathbb{X}} := \{x \quad s.t. \quad \psi(q(x)) \leq \check{\ell}_0\} \quad (39)$$

Figure 3.(b) shows the corresponding curves for q_{min} where the solid curve represents the r.h.s of (35) that tangents from below the solid curve of Figure 3.(a)

Now assume that q_{min} is found such that (35) holds, would this inequality be necessarily satisfied for $q \geq q_{min}$? The following Proposition answers this question:

Proposition 4.3. If the following inequality holds:

$$q_{max}(\tau_c) := \frac{D}{\gamma_1}\bar{m}(\check{\epsilon}_0,\tau_c) > q_{min} \quad (40)$$

then the inequality:

$$\check{\epsilon}_0 + K_0(E_1\bar{m}(\check{\epsilon}_0,\tau_c)) < \Gamma(\bar{m}(\check{\epsilon}_0,\tau_c),q) - \gamma_1\psi(q) \quad (41)$$

holds for all $q \in [q_{min}, q_{max}(\tau_c)]$ ♢

Proof. Since the inequality (41) holds for $q = q_{min}$, it is sufficient to prove that the r.h.s of (41) is monotonic non decreasing function on the interval $[q_{min}, q_{max}(\tau_c)]$. To do so, note that the derivative of the r.h.s satisfies:

$$\frac{\partial}{\partial q}\left[\Gamma(\bar{m}(\check{\epsilon}_0,\tau_c),q) - \gamma_1\psi(q)\right] \geq \begin{cases} \bar{m}(\check{\epsilon}_0,\tau_c) - \gamma_1\dfrac{q}{D} & \text{if } N \leq q/D \\ 0 & \text{otherwise} \end{cases} \quad (42)$$

which is obviously nonnegative whenever $q \leq q_{max}(\tau_c)$ given by (40). □

Figure 3.(c) shows both side of the inequality (34) for some $q > q_{min}$ in the case where inequality (40) of Proposition 4.3 holds. It clearly shows that for any $q \in (q_{min}, q_{max}(\tau_c)]$, the range of values of ϵ_0 that satisfies at x, the certification conditions of Proposition 4.1 are those lying within the following two **state-dependent** bounds:

$$\underline{\epsilon}_0(q(x)) \leq \epsilon_0 \leq \bar{\epsilon}_0(q(x)) := \min\{\tilde{\epsilon}_0(q(x)), \gamma_1\gamma_2\psi(q(x))\}$$

One of the implications of this result is that when asymptotic algorithms, in the sense of definition 3.1, are used (such as the fast gradient for instance), the lower bound on the precision, namely $\underline{\epsilon}_0(q(x))$ is strictly positive meaning that moderate precision is better than excessive precision.

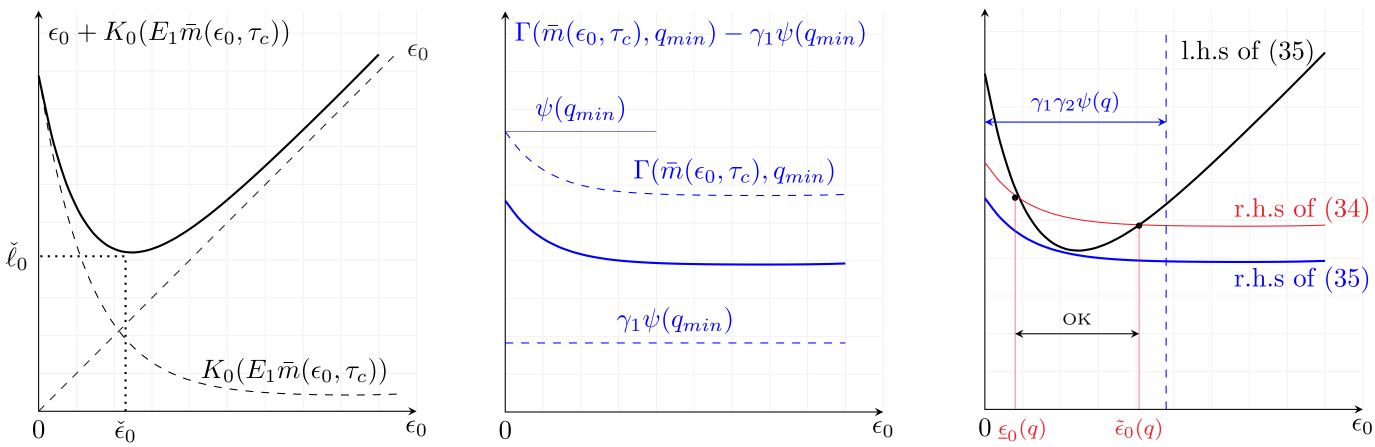
Another implication is that one can derive an off-line computed state-dependent control updating period. Indeed, one can adopt the general expression:

$$\epsilon_0(q(x)) = \lambda\left[\underline{\epsilon}_0(q(x))\right] + (1-\lambda)\left[\bar{\epsilon}_0(q(x))\right] \quad (43)$$

where $\lambda \in (0,1)$ can be taken close to 0 in order to enhance the highest control updating rate. Typical variation of the resulting state-dependent precision is shown in Figure 4 where the two bounds $\underline{\epsilon}_0(q(x))$ and $\bar{\epsilon}_0(q(x))$ are plotted vs $q(x)/q_{min}$. As expected, when $q(x) = q_{min}$ only a single solution exists (which is precisely $\check{\epsilon}_0$ invoked above while the interval of possible values enlarges when $q(x)$ increases enabling a decrease in the cost function to take place despite low precision (high values of ϵ_0) far from the terminal set defined by (32).

Remark 7. In (Richter et al. (2012)) an interesting question regarding the definition of the desired absolute precision ϵ_0 on the cost function is raised. The argument is that one can multiply the cost function by any factor without changing the problem while this obviously increases the number of iterations if the same precision is maintained. One may therefore ask: How does this *apparent* paradox is solved with the proposed scheme that does use absolute precision assignment?

The answer can be found by a careful examination of Figure 3.(c) which shows that if the cost function is multiplied by some factor, then the red curve representing the r.h.s of (34) will move up (or down) by the same factor. Fortunately, this also shifts the upper bound $\bar{\epsilon}_0(q(x))$ of admissible precision to the right by (asymptotically) the same amount (this is because the asymptotic upward tends



(a) l.h.s of (34) (b) r.h.s of (35) (c) The inequality (34)

Fig. 3. Decomposition of the main inequality (27) involved in Proposition 4.1. Definition of the two state-dependent bounds $\underline{\epsilon}_0(q(x))$ and $\bar{\epsilon}_0(q(x)) = \min\{\tilde{\epsilon}_0(q), \gamma_1\gamma_2\psi(q)\}$.

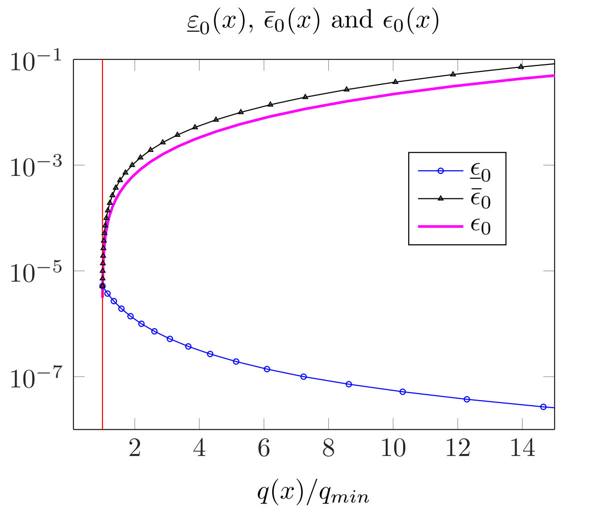


Fig. 4. Typical evolution of the state-dependent targeted precision $\epsilon(x)$ leading to the state-dependent control updating period $\tau(x) := [\bar{m}(\epsilon_0(x), \tau_c)]\,\tau_s$. Example of a chain of integrator system (Alamir (2015)).

towards identity).

In other words, the paradox disappears by the state-dependent nature of the absolute precision (and hence of the control updating period) while the question is legitimate in a state-independent setting. ⋄

5. EXAMPLE: LINEAR MPC WITH INPUT CONSTRAINTS

5.1 General Settings

Consider the case of LTI system given by:

$$\dot{z} = A_c z + B_c u + G_c w \tag{44}$$

for which the discrete-time version can be written using the sampling period τ_s as follows:

$$z^+ = Az + Bu + Gw \quad ; \quad y = Cz \in \mathbb{R}^{n_y} \tag{45}$$

with the control objective consisting in tracking the desired steady state defined by $z_d = S_1 y_d + S_2 w$. This can be done using the extended state $x := (z, y_d, w)$ and the presumed dynamic $y_d^+ = y_d$ and $w^+ = w$ on the exogenous signals y_d and w whose dynamics are supposed to be unknown. The following cost function can then be used:

$$J(\mathbf{u}_k, x_k) := \sum_{i=1}^{N_p} \|Cx^{\mathbf{u}}_{k+i|k}\|^2_Q + \|u_{k+i-1}\|^2_R \tag{46}$$

where $C := (\mathbb{I}_n \; -S_1 \; -S_2)$ corresponds to the tracking error matrix while Q and R are appropriate weighting matrices. Assume that the following restrictions hold:

$$u \in [-\bar{u}, \bar{u}] \quad ; \quad w \in [-\bar{w}, \bar{w}] \quad ; \quad y_d \in [-\bar{y}_d, \bar{y}_d] \tag{47}$$

In order to enforce the stability of the ideal MPC, an equality constraint on the final state is imposed, namely $Cx^{\mathbf{u}}_{k+N_p|k} = 0$ which can obviously by written using standard transition matrices Φ_{N_p} and Ψ_{N_p} as follows:

$$\big[C\Phi_{N_p}\big]\, x_k + \big[C\Psi_{N_p}\big]\,\mathbf{u} = 0 \tag{48}$$

and using the following parametrization of the control profile:

$$\mathbf{u} = K\mathbf{p} + M\mathbf{v} \quad \text{where } K := Ker(C\Psi_{N_p}) \tag{49}$$

the stabilizing constraint (48) becomes:

$$\big[C\Phi_{N_p}\big]\, x_k + \big[C\Psi_{N_p} M\big]\,\mathbf{v} = 0 \tag{50}$$

that explicitly gives $\mathbf{v}$ leading to the following stabilizing reduced order parametrization of the control profile:

$$\mathbf{u} = K\mathbf{p} + Tx_k \tag{51}$$

which satisfies the equality constraint $Cx^{\mathbf{u}}_{k+N_p|k} = 0$ for any choice of the remaining d.o.f contained in $\mathbf{p}$. Now obviously, the equality (51) cannot be satisfied under saturated control for any N_p and any x_k. Assuming in the sequel that N_p is fixed, this constraint can be imposed for those initial states x_k such that:

$$-\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \otimes \bar{u} \le Tx_k \le \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \otimes \bar{u} \tag{52}$$

The compact set of such x_k is denoted hereafter by $\mathbb{X}(\bar{u})$. Note that this defines a restriction on both the physical state of the system z as well as the set-point amplitude y_d and the disturbance level w. To summarize, given any $x_k \in \mathbb{X}(\bar{u})$, by choosing $\mathbf{p} = 0$ in (51), one gets a nominal trajectory that satisfies the final stability constraint on the state of the extended system.

A direct consequence of the above MPC formulation including the final equality constraint is that Assumption 3.4 holds with the following definition of $q(x)$:

$$q(x) = \|Cx\|_Q^2 \tag{53}$$

This together with (44) enables to compute an upper bound on the speed with which q can be steered to 0 since:

$$|\dot{q}(x)| \leq \max_{(x,u)|(44),\,(47),\,(52)} |2x^T C^T Q C\dot{x}| \tag{54}$$

which means that Assumption 3.5 holds with the D defined by (54).

The prediction error parameter E_1 invoked in Assumption 3.3 is obviously given by:

$$E_1 := (\max_{\ell=1}^{N_p} \|\Psi_\ell^G\|)\|\bar{w}\| + \delta_w^{max} + \delta_y^{max} \tag{55}$$

where δ_w^{max} and δ_y^{max} are the maximum increments on w and y_d that may occur in a single sampling period τ_s while Ψ_ℓ^G is given by:

$$\Psi_\ell^G := \begin{pmatrix} A^{\ell-1}G & \dots & AG & G \end{pmatrix} \tag{56}$$

Note that the original cost (46) together with the control parametrization (51) enables the cost to be rewritten in the new decision variable p in the following form:

$$J(\mathbf{p}, x_k) := \frac{1}{2}\mathbf{p}^T H\mathbf{p} + (F_0 + F_1 x_k)\mathbf{p} \tag{57}$$

The saturation constraints on the input becomes because of (51)

$$\underbrace{-\begin{pmatrix}\bar{u}\\ \vdots \\ \bar{u}\end{pmatrix} - Tx_k}_{g^{min}(x_k) :=} \leq K\mathbf{p} \leq \underbrace{\begin{pmatrix}\bar{u}\\ \vdots \\ \bar{u}\end{pmatrix} - Tx_k}_{:= g^{max}(x_k)} \tag{58}$$

Unfortunately, when it comes to derive certification bounds on optimization algorithms (such as the fast gradient we consider in the sequel), only the case of simple bounds on the decision variables can be considered. That is the reason why (58) is transformed into a simple bounds set of constraints at the price of introducing some level of sub-optimality. This is done by first observing that the constraints (58) hold for $\mathbf{p} = 0$ (by construction). Therefore one can satisfy (58) provided that the following simple bound constraints are satisfied:

$$\underline{p}_j \leq p_j \leq \bar{p}_j \tag{59}$$

in which the new bounds are defined by:

$$\underline{p}_j := \max\Bigl\{ \max_{i, K_{ij}<0} \frac{g_i^{max}}{n_p K_{ij}}, \max_{i, K_{ij}>0} \frac{g_i^{min}}{n_p K_{ij}} \Bigr\} \tag{60}$$

$$\bar{p}_j := \min\Bigl\{ \min_{i, K_{ij}>0} \frac{g_i^{max}}{n_p K_{ij}}, \min_{i, K_{ij}<0} \frac{g_i^{min}}{n_p K_{ij}} \Bigr\} \tag{61}$$

which, together with (57) leads to the constant $K_0 := \|F_1\| \max\{\|\bar{p}\|, \|\underline{p}\|\}$ invoked in (16) of Assumption 3.6. Regarding the initial guess $\mathbf{u}_k^{init}$ invoked in Assumption 3.1, the following development is done using the initial guess corresponding to $\mathbf{p} = 0$. The last result we need to proceed is to choose the optimization algorithm and the corresponding certification bound $N(\epsilon_0)$ invoked in Assumption 3.1. Note that no ϵ_c is involved since there is no soft constraints to be considered in our example. Using the Fast Gradient algorithm, the following certification bound can be used (Nesterov (2004), page 80):

$$N(\epsilon_0) := \max\left\{0, \left\{\frac{\log(\gamma(\epsilon_0))}{\log(1-c)}, \frac{1}{c}(\sqrt{\frac{1}{\gamma(\epsilon_0)}} - 1)\right\}\right\} \tag{62}$$

where:

$$c := [\lambda_{min}(H)/\lambda_{max}(H)]^{\frac{1}{2}} \tag{63}$$

$$\gamma(\epsilon_0) := \frac{2\epsilon_0}{(\lambda_{min}(H) + \lambda_{max}(H)) \cdot \|\bar{p} - \underline{p}\|^2} \tag{64}$$

5.2 Example: Chain of integrators

Consider a chain of $n = 3$ integrators given by:

$$\dot{z}_i = z_{i+1} \quad \text{for } i = 1, \dots, n-1 \tag{65}$$

$$\dot{z}_n = u \tag{66}$$

to which we add the disturbance through the matrices $G_c = (1, 0, 0)^T$. Consider the following saturation levels $\bar{u} = 5$, $\bar{w} = 0.01$ and the matrices S_1 and S_2 such that the steady state is given by the stationary state z_d that is compatible with some desired output $z_1 = y_d$. We consider the basic sampling period $\tau_s = 20ms$ and the prediction horizon length $N_p = 100$. These leads to the following quantities: $D = 0.7172$, $\|\bar{p} - \underline{p}\| = 1.04$, $K_0 = 0.97$, $\max_{\ell=1}^{N_p} \|\Psi_\ell^G\| = 0.34$, $c = 0.17$. The weighting matrix $Q = \mathbb{I}_3$ and $R = 0.1$ are used in the definition of the cost function (46). The computation time τ_c needed for a single iteration of the fast gradient algorithm is taken equal to $\tau_c = 10\mu$s. The maximum derivatives of y_d and w that are used to compute δ_w^{max} and δ_y^{max} involved in the computation of E_1 through (55) are both set to 0.1. This gives everything necessary to plot the curves shown in Figure 3 in the general case for the specific case at hand.

Figure 5 shows the satisfaction of the fundamental inequality (35) with $q_{min} = 0.051$ and $\gamma_1 = 0.05$. This Figure is similar to the general Figure 3.(c)in the specific case of the triple integrator [$n = 3$ in (65)]. Using this Figure, one can compute the two parameters $\check{\epsilon}_0 = 5.89 \times 10^{-4}$ and $\check{\ell}_0 \approx 0.0018$ shown in Figure 3.(a) and invoked in (40) and (38) respectively. Now using $\gamma_2 = 0.95$ one can compute another lower bound on q that makes condition (28) of Proposition 4.1 satisfied, more precisely:

$$\psi(q_{min}) \geq \max\left\{\psi(0.051), \frac{\check{\epsilon}_0}{\gamma_1\gamma_2}\right\} \tag{67}$$

which leads to $q_{min} = 0.124$.

Figure 6 shows the resulting bounds on the state-dependent precision to be used in the truncated optimization process as a function of the level set $q(x)$. This message of Figure 6 is very interesting as it tells that given the current value of the state, and hence of $q(x)$, the successful choice of the required precision in solving the optimization problem lies between a lower and an upper bounds. Any choice that lie beyond this interval increases

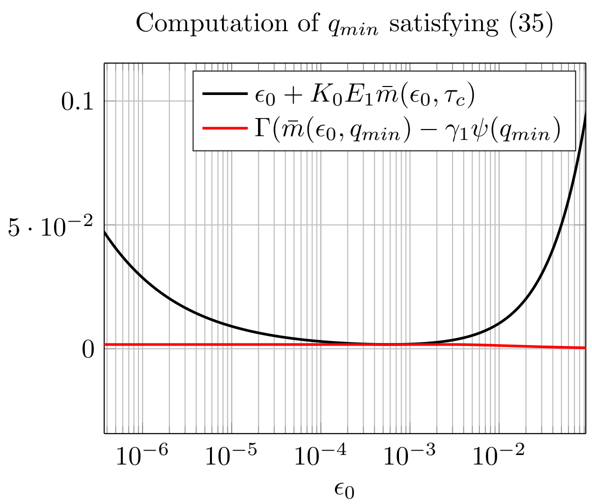


Fig. 5. Instantiation of Figure 3.(c) in the case of the triple integrator example. Satisfaction of the fundamental inequality (35) for $q_{min} = 0.051$. The computation time $\tau_c = 10\mu$s is considered for a single iteration of the fast gradient algorithm.

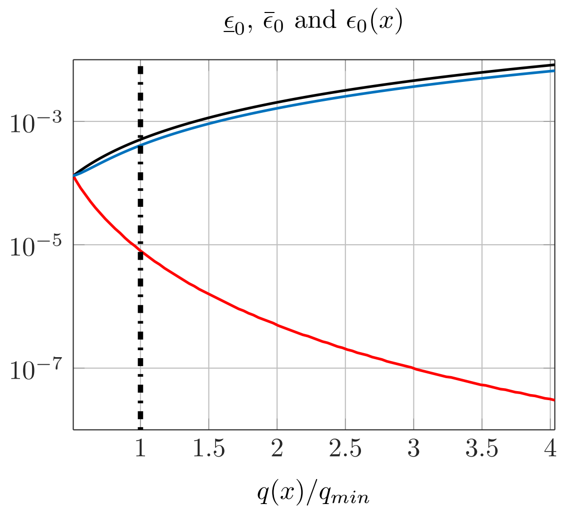


Fig. 6. Resulting state dependent targeted precision $\epsilon_0(x)$ for the triple integrator example with $\tau_c = 10\mu$s.

the size of the final region.

It is important to note that any change in the MPC design parameter leads to different certification results. In particular, the use of shorter prediction horizon $N_p = 50$ (instead of 100) leads to a certified terminal set that is defined by $q_{min} = 0.048$ (instead of 0.124). Note however that this is obtained at the price of smaller admissible initial state as the final constraint [see (48) and (52)] would be harder to achieve on a shorter prediction horizon.

6. CONCLUSION AND FUTURE WORK

In this paper, a certification framework is proposed for general MPC schemes with provable stability ingredients and an optimizer for which a certification bound is available. The framework addresses the case where prediction errors are present either because of model mismatch or because of unknown set-point dynamics or both. The general setting is then described in details in the case of linear time invariant system tracking time varying signal under the presence of uncertainties.

An obvious continuation of this work is to rationalize the choice of the targeted precision $\epsilon_0^{(i+1)}$ to be chosen inside the admissible interval depicted in Figure 6. Indeed, while any choice inside the admissible range does lead to guaranteed decrease of the cost function, there is obviously a specific choice that increases the rate of convergence. This choice is not obvious to be made beforehand because the decreasing property is only certified through worst case analysis. The question is to which extent the heuristics proposed in Alamir (2013) and Alamir (2014) can be made more formally assessed.

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