# Re-injecting the structure in NMPC schemes

## Application to the constrained stabilization of a snakeboard

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**Summary.** In this paper, a constrained nonlinear predictive control scheme is proposed for a class of under-actuated nonholonomic systems. The scheme is based on fast generation of steering trajectories that inherently fulfill the contraints while showing a "*translatability*" property which is generally needed to derive stability results in receding-horizon schemes. The corresponding open-loop optimization problem can be solved very efficiently making possible a real-time implementation on fast systems (The resulting optimization problem is roughly scalar). The whole framework is shown to hold for the well known challenging problem of a snakeboard constrained stabilization. Illustrative simulations are proposed to assess the efficiency of the proposed solution under saturation constraints and model uncertainties.

## 1 Introduction

One of the most attractive features of Nonlinear Model Predictive Control (**NMPC**) schemes [1, 2] is their complete independence of the mathematical structure of the system's model. Indeed, from a conceptual point of view, given any system satisfying a rather intuitive set of assumptions, one may write down a concrete state feedback algorithm that theoretically asymptotically stabilizes a target equilibrium state.

Unfortunately, such generically defined formulations may lead to optimization problems that cannot be solved in the available computation time when rather fast dynamics are involved. This can be formulated in a kind of "*no free lunch*" statement :

Genericity reduces efficiency

Therefore, to overcome the consequences of the above unavoidable truth, specific features of each system under study have to be explicitly taken into account as far as possible.

In a series of papers [3, 4, 5, 6, 7, 8, 9], it has been shown that when the constrained stabilization is the main issue, that is, when the optimality is not rigorously required, efficient stabilizing **NMPC** schemes can be obtained as soon as open-loop "*steering trajectories*" can be generated by some systematic and efficient algorithm.

Now, the way these trajectories are generated is system dependent and the associated efficiency may be greatly increased if the specific structure of the system is explicitly exploited. This is what reduces the genericity to increase efficiency. By allowing low dimensional parametrization of these trajectories, corresponding low dimensional **NMPC** schemes can be defined in which, the decision variable is the parameter vector of the steering trajectory.

In this paper, it is shown that for a particular class of mechanical systems including the snakeboard, it is possible to use the particular structure of the system equations in order to derive efficient computation of parameterized steering trajectories. Moreover, these trajectories have the nice property of being structurally compatible with the saturation constraints on the actuators. Since in addition, they have the "*translatability property*", they can be used to implement a stable closed loop receding horizon feedback.

The paper is organized as follows: First, the particular class of mechanical systems under study is defined in section 2 together with the associated assumptions. In section 3, the proposed state feedback algorithm is explained and the associated convergence results are derived. The fact that the snakeboard falls into the particular class depicted in section 2 is discussed in section 4. Finally, illustrative simulations are proposed in section 5 in order to assess the performance of the proposed solution.

## 2 The class of systems considered

We consider nonlinear systems that may be described by the following set of ODE's

$$\dot{r} = f_1(\chi)g_1(\eta)\dot{\eta} \tag{1}$$

$$\dot{\eta} = f_2(\chi)g_2(\xi) \quad ; \quad g_2(0) = 0 \tag{2}$$

$$\dot{\xi} = f_3(\xi, \chi, \dot{\chi}, u_1) \tag{3}$$

$$\ddot{\chi} = f_4(\xi, \chi, \dot{\chi}, u_2) \tag{4}$$

where equations (1)-(2) stands for a KINEMATIC stage while equations (3)-(4) represent the DYNAMIC stage. $r \in \mathbb{R}^{n_r}$ is a kind of generalized position; $\eta \in \mathbb{R}^{n_\eta}$ is an orientation variable; $\xi \in \mathbb{R}^{n_\xi}$ is a generalized velocity while $\chi \in \mathbb{R}$ stands for an internal configuration variable. $f_1 : \mathbb{R} \to \mathbb{R}$, $g_1 : \mathbb{R}^{n_\eta} \to \mathbb{R}^{n_r \times n_\eta}$. All the maps invoked in (1)-(4) are assumed to be continuously differentiable.

Note that equation (1) generally describes a **nonholonomic** constraint, namely, a constraint on the velocities that is not derived from a position related constraints. The control inputs $u_1$ and $u_2$ have to meet the following saturation constraints

$$\forall t \quad , \quad u_i(t) \in [-u_i^{max}, +u_i^{max}] \quad ; \quad i \in \{1, 2\} \tag{5}$$

for some given upper bounds $u_1^{max}$ and $u_2^{max}$.

The aim of the present paper is to derive a state feedback algorithm that steers the sub-state $(r, \eta, \xi)$ of the system (1)-(4) to the origin under the saturation constraint (5). Note that since $g_2(0) = 0$, the origin $(r, \eta, \xi) = 0$ is an equilibrium position for the dynamics (1)-(3) provided that $\xi$ can be maintained at 0 by convenient use of the control input $u_1$ as it is suggested by Assumption 1 hereafter. Note also that $\chi$ is an internal configuration variable whose value is irrelevant for the control objective. This is for instance the angular position of the wheels in the snakeboard example (see section 4).

Beside the structure depicted in equations (1)-(4), the class of systems of interest has to satisfy the following assumptions :

**Assumption 1**

1. *For all $\chi^f$, there exists a feedback law*

$$u := (u_1, u_2) = K_1(\xi, \chi, \dot{\chi}, \chi^f)$$

*under which, the closed loop behavior respects the constraints (5) and such that*

$$(\xi = 0, \dot{\chi} = 0, \chi = \chi^f)$$

*is a globally asymptotically stable equilibrium for the closed loop dynamics defined by (3)-(4) and $K_1$. Furthermore, the subset $\{\xi = 0\}$ is invariant under the closed loop behavior.*

2. *For all $\chi^f \neq 0$ and all $\eta^f$, there exists a feedback law*

$$u = (u_1, u_2) = K_2(\eta, \xi, \chi, \dot{\chi}, \eta^f, \chi^f)$$

*such that the closed loop behavior respects the constraints (5) and such that*

$$(\eta = \eta^f, \xi = 0)$$

*is a globally asymptotically stable equilibrium for the closed loop dynamics defined by (2)-(3) and $K_2$. Furthermore, the set $\{\chi = \chi^f, \dot{\chi} = 0\}$ is invariant under this dynamics.*

3. *For all $\eta^0$, there exists a parameterized sequence $(\eta^k(p, \eta^0))_{k\geq 0}$ (defined for some parameter vector $p \in \mathbb{P}$ where $\mathbb{P}$ is a compact set) that is continuous in $(p, \eta^0)$ such that $\eta^0(p, \eta^0) = \eta^0$, $\lim_{k\to\infty} \eta^k(p, \eta^0) = 0$ (**exponentially**) and the following rank condition is satisfied for all $p \in \mathbb{P}$:*

$$Rank\Big[A(p, \eta^0) = \big(A_1(p, \eta^0) \ \dots \ A_j(p, \eta^0) \ \dots\big)\Big] = n_r \tag{6}$$

*where*

$$A_j(p, \eta^0) = \int_{\eta^{j-1}(p,\eta^0)}^{\eta^j(p,\eta^0)} g_1(\eta) d\eta \tag{7}$$

*with $\sigma_{min}(A(p, \eta^0)) > s_{min} > 0$. Furthermore, for all $p \in \mathbb{P}$, there exists $p^+ \in \mathbb{P}$ such that*

$$\eta^k(p^+, \eta^1(p, \eta^0)) = \eta^{k+1}(p, \eta^0) \quad \forall k \geq 0 \quad \text{(translatability)} \tag{8}$$

4. *The function $g_2$ in (2) is such that for any pair $(\eta^a, \eta^b)$, there exists a sequence*

$$(\eta_j^{(a,b)})_{j=1}^{n_r} \quad ; \quad \eta_1^{(a,b)} = \eta^a \quad ; \quad \eta_{n_r}^{(a,b)} = \eta_b$$

*such that the matrix*

$$M(\eta^a, \eta^b) := \left( \int_{\eta_1^{(a,b)}}^{\eta_2^{(a,b)}} g_1(\eta)d\eta \ldots \int_{\eta_{n_r-1}^{(a,b)}}^{\eta_{n_r}^{(a,b)}} g_1(\eta)d\eta \right) \in \mathbb{R}^{n_r \times n_r} \tag{9}$$

*has full rank.*

5. *The function $f_1$ maps some domain $\mathcal{D} \subset \mathbb{R}$ onto $\mathbb{R} - \{0\}$ so that an inverse map $f_1^{-1} : \mathbb{R} - \{0\} \to \mathcal{D}$ may be defined by fixing some selection rule (when needed).*

In the following sequel, the state of the system is denoted by $x$, namely

$$x := \begin{pmatrix} r \; \eta \; \xi \; \chi \; \dot{\chi} \end{pmatrix}^T \in \mathbb{R}^n \quad ; \quad n = n_r + n_\eta + n_\xi + 2$$

## 3 The proposed feedback algorithm

The basic idea of the control algorithm is to decompose the behavior of the controller into basically 2 modes

1. In the first mode, $\xi \approx 0$ and the feedback $K_1$ is used to steer the state $\chi$ from some initial value $\chi_{j-1}$ to some final desired one $\chi_j$. This is possible thanks to assumption 1.1. Note that under this mode the position $r$ as well as the orientation variable $\eta$ are maintained almost constant (since $\xi \approx 0$ and $g_2(0) = 0$ by assumption 1.5).
2. In the second mode, $\chi \approx \chi_j$ is maintained almost constant while the feedback $K_2$ is used to steer the variable $\eta$ from some initial value $\eta^{j-1}$ to some final value $\eta^j$. Note again that this is possible thanks to assumption 1.2. Note moreover that under this mode, $\xi$ asymptotically goes to 0 enabling the first mode to be fired again.

Note that from equations (1) and (7), it comes that under constant $\chi \equiv \chi_j$, when the mode 2 is used with $\eta$ changing from $\eta^{j-1}(p, \eta^0)$ to $\eta^j(p, \eta^0)$, the corresponding variation in $r$ is given by :

$$\Delta_j r := \Big[ A_j(p, \eta^0) \Big] f_1(\chi_j) \quad \text{under constant } \chi \equiv \chi_j \text{ (mode 2)} \tag{10}$$

Therefore, the condition

$$r^0 + A(p, \eta^0) \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_j \\ \vdots \end{pmatrix} = 0 \quad ; \quad v_j \neq 0 \quad \text{for all } j \quad ; \quad \chi_j = f_1^{-1}(v_j) \tag{11}$$

characterizes the family of sequences $(\chi_j)_{j \geq 0}$ such that when the two-modes procedure defined above is applied in an open-loop way, the vector $(r, \eta, \xi)$ is steered to 0.

The state feedback proposed in the present paper amounts to use the open-loop strategy defined above in a receding horizon way. This is because the sequences $(\eta^j)_{j\geq 0}$ and $(\chi_j)_{j\geq 0}$ may become irrelevant because of unavoidable disturbances and because of the simple fact that during mode 1 [resp. mode 2], $\xi = 0$ [resp. $\chi = \chi_j$] cannot be rigorously satisfied making necessary to re-compute the steering strategy.

In order to properly define the proposed receding horizon formulation, the following definitions are needed :

**Definition 1.** *Given any $(p, r^0, \eta^0)$, let $q \in \mathbb{N}$ be a sufficiently high integer for $A_j(p, \eta^0)$ to be negligible for $j > q$ and define the vector $\hat{v}(p, r^0, \eta^0) \in \mathbb{R}^q$ to be the solution of the following Linear Programming (LP) problem:*

$$\hat{v}(p, r^0, \eta^0) := \text{Arg} \min_{\tilde{v} \in \mathbb{R}^q} \Big[ -\sum_{j=1}^{q} \tilde{v}_j \Big]$$
$$\text{under } r^0 + \big[A_{1\to q}(p, \eta^0)\big]\tilde{v} = 0 \text{ and } \|\tilde{v}\|_\infty \leq v_{max} \tag{12}$$

*where $A_{1\to j}(p, \eta^0)$ is the matrix built with the $j$ first matrix terms of $A(p, \eta^0)$ [see (6)] while $v_{max}$ is a sufficiently high value making the above constrained problem feasible for any initial value $r^0$ of interest.*

Note that for each candidate value of $p \in \mathbb{P}$, the LP problem (12) may be solved almost instantaneously using LP solvers. Note also that the cost function to be minimized suggests solutions that avoid vanishing components of $\hat{v}$. However, if in spite of this $\hat{v}_k(p, r^0, \eta^0) = 0$ for some $k$, intermediate values have to be introduced in the sequence $(\eta^j(p, \eta^0))_{j\geq 0}$ between $\eta^k(p, \eta^0)$ and $\eta^{k+1}(p, \eta^0)$ in order to remove this vanishing components without altering the remaining solution, namely $\hat{v}_i(p, r^0, \eta^0)$ for $i \geq k + 1$. This is possible thanks to assumption 1.4. Indeed :

- Assume that for some $k$, one has $\hat{v}_k = 0$, this means that the positions $r$ at the sampling instants $k$ and $k + 1$ are the same.
- Consider the following matrices defined according to (9) :

$$M(\eta^k, \eta^{k+\frac{1}{2}}) \quad ; \quad M(\eta^{k+\frac{1}{2}}, \eta^{k+1}) \quad \text{where} \quad \eta^{k+\frac{1}{2}} := \frac{\eta^k + \eta^{k+1}}{2}$$

  together with the corresponding sequences

$$\Big(\eta_j^{k,k+\frac{1}{2}}\Big)_{j=1}^{n_r}, \Big(\eta_j^{k+\frac{1}{2},k+1}\Big)_{j=1}^{n_r}$$

- Our aim is to prove that there exists a sequence of controls

$$\Big(\hat{v}_j^k\Big)_{j=1}^{2n_r}$$

  with no zero elements and such that the net variation on $r$ vanishes. But this amounts to find a vector $v \in \mathcal{C}_p := [-v_{max}, v_{max}]^{2n_r}$ with $v_i \neq 0$ for all $i$ such that the vector

$$\underbrace{\left[M(\eta^{k+\frac{1}{2}},\eta^{k+1})\right]^{-1}\left[M(\eta^{k},\eta^{k+\frac{1}{2}})\right]}_{A}v \tag{13}$$

has no zero elements. But this is clearly always possible since the matrix $A$ is regular according to assumption 1.4. (See lemma 1 in the appendix for a brief proof of this evident fact).

Repetitive application of this technique enables all vanishing (or too small) components of $\hat{v}$ to be removed.

**Definition 2.** *Given any $(r^0,\eta^0)$, the optimal parameter vector $\hat{p}(r^0,\eta^0)$ is defined by*

$$\begin{aligned}\hat{p}(r^0,\eta^0) &:= \text{Arg}\min_{p\in\mathbb{P}} J(p,r^0,\eta^0)\\ &:= \sum_{j=0}^{q}\Big[\|r^0 + A_{1\to j}(p,\eta^0)\hat{v}_{1\to j}(r^0,\eta^0)\|^2 + \alpha\cdot|\eta^j(p,\eta^0)|^2\Big]\end{aligned} \tag{14}$$

*namely, $\hat{p}(r^0,\eta^0)$ minimizes a quadratic cost on the excursion of the configuration vector $(r,\eta)$.*

Note that definition 2 assumes that the admissible parameter set $\mathbb{P}$ is such that (14) admits a solution. This is typically guaranteed because of the continuity of the cost function $J$ w.r.t. $p$ and the compactness of the admissible set of parameters $\mathbb{P}$.

Putting together definitions 1 and 2 enables us to define for any initial configuration $(r^0,\eta^0)$ the optimal sequence given by

$$v^{opt}(r^0,\eta^0) := \hat{v}(\hat{p}(r^0,\eta^0),r^0,\eta^0) = \big(v_1^{opt}(r^0,\eta^0)\ \dots\ v_q^{opt}(r^0,\eta^0)\big) \in \mathbb{R}^q \tag{15}$$

and since according to the discussion that follows definition 1, one can assume without loss of generality that $v_j^{opt}(r^0,\eta^0)\neq 0$, the following may be defined for a given pair $(r^0,\eta^0)$ thanks to assumption 1.5 :

$$\hat{\eta}(r^0,\eta^0) := \eta^1(\hat{p}(r^0,\eta^0),\eta^0) \quad ; \quad \hat{\chi}(r^0,\eta^0) := f_1^{-1}\Big(v_1^{opt}(r^0,\eta^0)\Big) \tag{16}$$

which is nothing but the first part of the OPTIMAL open loop trajectory on $(\eta,\chi)$ mentioned above when used in a receding horizon way. Finally, in order to monitor the selection of the controller's mode, the following functions are needed :

$$V_1(x,\chi^f) = \max\Big\{|\chi-\chi^f|,|\dot{\chi}|,|\xi|\Big\} \quad ; \quad V_2(x,\eta^f) := \max\Big\{|\eta-\eta^f|,|\xi|\Big\} \tag{17}$$

More precisely, when $V_1(x,\mathcal{X}^f)$ approaches 0, this means that the controller task at mode 1 is likely to be achieved. Similarly, when $V_2(x,\eta^f)$ approaches 0, this means that the controller task in mode 2 is not far from being achieved. Now, it goes without saying that one cannot wait for $V_1$ or $V_2$ to be exactly equal to zero since this never happens rigorously. That is the reason why a finite threshold $\varepsilon>0$ is used in the definition of the switching rules given hereafter.

Using the above notations, the proposed state feedback algorithm can be expressed as follows

FEEDBACK ALGORITHM

**parameters** $\varepsilon > 0$ a small threshold. $v_{max} > 0$ sufficiently large value to be used in (12)
**1)** Compute $\chi^f = \hat{\chi}(r(t), \eta(t))$ and $\eta^f = \hat{\eta}(r(t), \eta(t))$ according to (16)
**2) mode 1**
**Use** the feedback $K_1$ with $\chi^f$ as computed in step **1) until** $V_1(x, \chi^f) \leq \varepsilon$.
**3) mode 2**
**Use** the feedback $K_2$ with $\eta^f$ and $\chi^f$ as computed in step **1) until** $V_2(x, \eta_f) \leq \varepsilon$
**4) go to step 1)**

As a matter of fact, the feedback algorithm presented above describes a Finite State Machine that is depicted in figure 1. Associated to the proposed feedback algorithm,

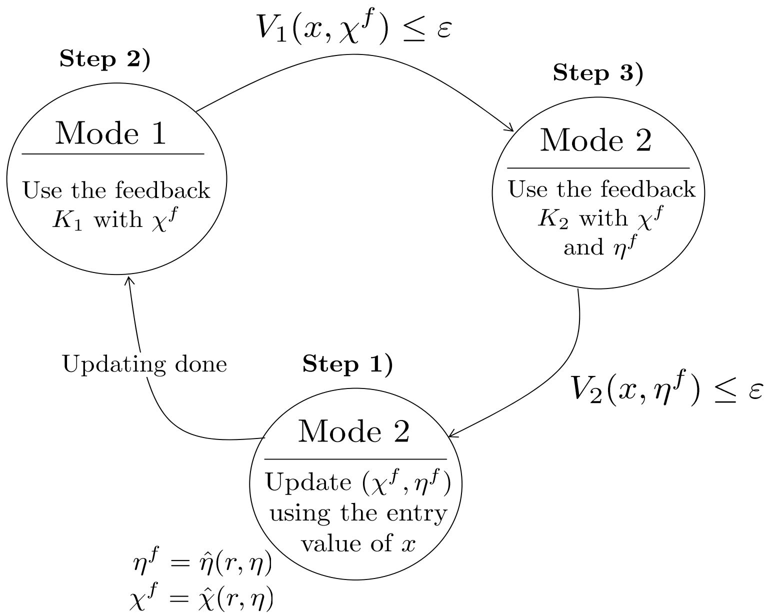


**Fig. 1.** The Finite State Machine describing the proposed feedback algorithm

the following convergence result can be proved

**Proposition 1.** *Let $(t_k)_{k \geq 0}$ be the infinite sequence of instants at which the algorithm visits the updating state of* **Step 1** *(see figure 1). We have the following asymptotic property*

$$\lim_{\varepsilon \to 0} \left[ \lim_{k \to \infty} \left\| \begin{pmatrix} r(t_k) \\ \eta(t_k) \\ \xi(t_k) \end{pmatrix} \right\| \right] = 0 \tag{18}$$

*namely, by taking* $\varepsilon$ *sufficiently small, the sequence* $(r(t_k), \eta(t_k), \xi(t_k))$ *may be steered as close as desired to* 0.

SKETCH OF THE PROOF The regularity assumption on the functions appearing in (1)-(4) together with the rank condition and the uniform regularity of the matrix $A(p, \eta^0)$ used in (12) enables to prove the continuity of the optimal value function $\hat{J}(r^0, \eta^0) = J(\hat{p}(r^0, \eta^0), r^0, \eta^0)$ w.r.t its arguments $r^0$ and $\eta^0$. Using this property with the translatability assumption (8) on the parameterization being used enables to write a classical inequality in receding horizon analysis, namely

$$\hat{J}(r(t_{k+1}), \eta(t_{k+1})) - \hat{J}(r(t_k), \eta(t_k)) \leq -\|r(t_k)\|^2 - \alpha\|\eta(t_k)\|^2 + O(\varepsilon) \quad (19)$$

where the final term regroups all the second order effects due to the use of finite stay time in each mode ($\varepsilon > 0$) and the use of finite horizon $q$ in the equality constraint of (12). Finally, since by definition of the exit condition of **Step 3**, one necessarily has

$$|\xi(t_k)| \leq \varepsilon \quad (20)$$

the result clearly follows from (19)-(20).

## 4 Application to the constrained stabilization of a snakeboard

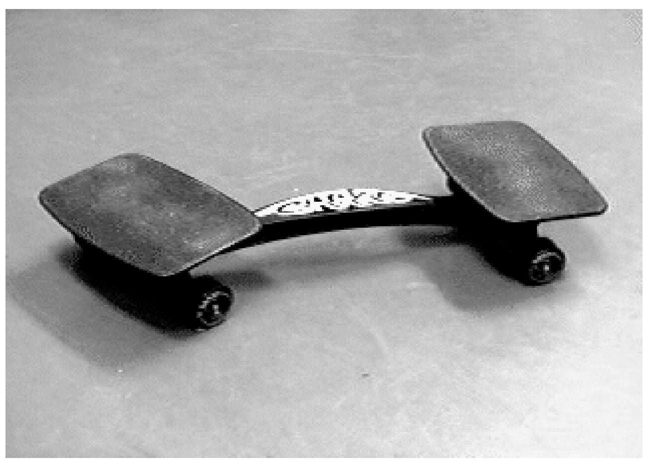

**Fig. 2.** The snakeboard "*without the rider*"

The snakeboard we are interested in is the mechanical system depicted in figures 3 and 2. It consists of two wheel-based platforms upon which a rider is to place each

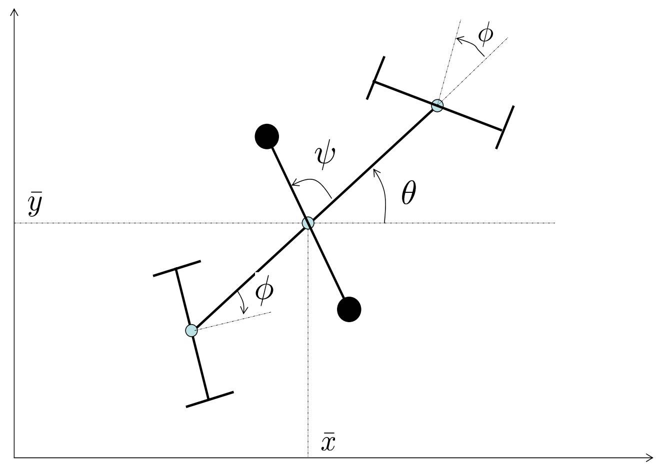


**Fig. 3.** Schematic view of the snakeboard's variables definition. Note that the front and the back steering angles are coupled to be the same

of his feet. These platforms are connected by a rigid coupler with hinges at each platform to allow rotation about the vertical axis. The snakeboard allows the rider to propel himself forward without having to make contact with the ground. This motion is obtained by using the conservation of angular momentum effect together with the nonholonomic constraints defined by the no-slipping condition at the wheels level.

This system was first outlined by [10] to be a particularly interesting nonholonomic system example that has no straightforward direct biological counterpart. In [10], a first dynamical model for the system was given and used to check some standard behaviors that correspond to some given oscillatory gaits. Furthermore, controllability analysis has been proposed showing that this system is locally controllable except at some singular configurations. The complete proof of controllability has been achieved in [11]. Since then, many works have been done to construct steering trajectories and control design. In [12], a strategy leading to the design of steering trajectories has been proposed based on small amplitude, short duration and cyclic inputs. Based on such sinusoidal inputs, a controller based on average theory has also been proposed recently [13].

Much closer to our approach that does not invoke cyclic open loop parametrization is the work proposed in [14]. Indeed, in [14] the generation of path steering trajectories is based upon switching between two vector fields in order to obtain sub-curves starting from the initial configuration and ending at the desired one. The system is at rest at the switching instants. The constraints handling is obtained by means of time scaling. As long as the snakeboard is concerned, the difference between the approach proposed in [14] and the one proposed in the present paper lies in the following differences:

- ✓ The way the sub-curves defined above are derived in [14] deeply depends on the $2D$ nature of the snakeboard example (intersections of circles and/or straight lines, etc. are extensively used). In our approach, this is basically done, even in the general case by solving linear systems. In that sense, the approach proposed here seems to be more easily generalizable, as long as the assumed structure of the system holds.
- ✓ The choice of the steering trajectories in [14] is based on minimizing the corresponding number of switches whatever is the resulting transient spacial excursion. The reason behind this is that the system has to be at rest at switching instants. Therefore, having a high number of switches may lead to a slow motion. In our case, monitoring the number of switches can be directly and explicitly obtained by the number ($q$) of intermediate values in the parameterized sequence :

$$\left(\eta^k(p,\eta^0)\right)_{k=1}^{q}$$

  while the additional d.o.f $p$ are used to minimize the corresponding spacial excursion in the $x-y$ plane.
- ✓ The work in [14] concentrates basically on the open-loop steering strategy. It is not completely clear whether the resulting steering strategy can be used in a receding horizon manner in order to yield an effective feedback in presence of uncertainties or modelling errors. In other words, the "*translatability*" of the open-loop trajectories proposed in [14] has to be checked. If some other feedback strategy is to be used, this is still to be designed.

The model of the snakeboard used in the forthcoming developments is given by :

$$\begin{pmatrix}\dot{\bar{x}}\\ \dot{\bar{y}}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2l\cos^2\phi\\ 0\\ \sin(2\phi)\end{pmatrix}\xi \tag{21}$$

$$\dot{\xi} = \left(-\frac{J_r}{2ml^2}\ddot{\psi} + \dot{\phi}\xi\right)\cdot\tan\phi \tag{22}$$

$$u_2 = 2J_w\ddot{\phi} \tag{23}$$

$$u_1 = J_r\Big[(1-\frac{J_r}{ml^2}\sin^2\phi)\ddot{\psi} + 2\dot{\phi}\xi\cos^2\phi\Big] \tag{24}$$

Note that the third line of Equation (21) writes:

$$\xi = \frac{\dot{\theta}}{\sin(2\phi)} = \frac{\dot{\theta}}{2\sin(\phi)\cos(\phi)}$$

Using this in the equations enables us to show that te snakeboard equations (21)-(24) are of the standard form (1)-(4) provided that the following correspondances are used

$$r = \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} \quad ; \quad n_r = 2 \quad ; \quad \eta = \theta \quad ; \quad n_\eta = 1 \quad ; \quad \chi = \phi \quad ; \quad f_1(\chi) = \frac{2l}{\tan\phi}$$

$$g_1(\eta) = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \quad ; \quad f_2(\chi) = \sin(2\phi) \quad ; \quad g_2(\xi) = \xi$$

with straightforward definitions of $f_3$ and $f_4$ that may be obtained by removing the auxiliary variable $\ddot{\psi}$. Note that in the classically used models [10, 14, 12], the configuration is given by $q = (\bar{x}, \bar{y}, \theta, \phi, \psi)$. Indeed, the variable $\xi$ used in the above equations is an intermediate variable that can be removed since from (21), it comes that

$$\xi = \frac{\dot{\bar{x}}}{2l\cos\theta\cos^2\phi} \tag{25}$$

which is clearly a function of $(q, \dot{q})$. However, in the derivation of the control law, writing the equation in the form (21)-(24) is mandatory in order to fit the standard form (1)-(4). This is a classical feature: when using partially structural approach, the coordinate system plays a key role.
In order to use the feedback scheme defined in section 3, the feedbacks $K_1$, $K_2$ and the parameterized sequence $\eta^k(p, \eta^0)$ are successively introduced.

### 4.1 Definition of the feedback laws $\mathbf{K_1}$ and $\mathbf{K_2}$

In both modes, the definition of $u_2$ is the same and is given by

$$u_2 := -2J_w\Big[\frac{2}{t_r}\dot{\phi} + \frac{1}{t_r^2}(\phi - \phi^f)\Big] \tag{26}$$

where $t_r$ is a time scaling, namely, for $t_r = 1$, a pole placement (with identical double poles $= -1$) is assigned by imposing

$$\ddot{\phi} = -2\dot{\phi} - (\phi - \phi^f)$$

The parameter $t_r$ is then used to meet the saturation requirement by taking :

$$t_r = \min_{\tau\in[0.01\ \ 10]} \tau \quad \text{such that} \quad 2J_w \cdot |\tfrac{2}{\tau}\dot{\phi} + \tfrac{1}{\tau^2}(\phi - \phi^f)| \le u_2^{max} \tag{27}$$

As for $u_1$, it is mode dependent, namely

$$u_1|_{\text{mode } i} := J_r\Big[(1 - \frac{J_r}{2ml^2}\sin^2\phi)\psi^{pp}|_{\text{mode } i} + 2\dot{\phi}\xi\cos^2\phi\Big] \quad ; \quad i \in \{1,2\} \tag{28}$$

$$\psi^{pp}|_{\text{mode } 1} := Sat^{\psi^{pp}_{max}(\phi,\dot{\phi},\xi)}_{\psi^{pp}_{min}(\phi,\dot{\phi},\xi)}\Big(10sign(\phi)\xi\Big) \tag{29}$$

$$\psi^{pp}|_{\text{mode } 2} := \frac{2ml^2}{J_r}\left(\frac{\mu}{\tan\phi}\Big[-0.1|sin2\phi|\xi + (\xi - \xi_r(\mu))\Big] + \dot{\phi}\xi\right) \tag{30}$$

where

- $\xi_r(\mu)$ is a varying reference value for $\xi$ that is given by

$$\xi_r := -0.1\mu \cdot sign(\sin 2\phi)(\theta - \theta^f) \tag{31}$$

- $\mu$ is an adaptive gain computed according to

$$\mu = \max_{\nu\in[0\ \ 100]} \nu \quad \text{such that}$$

$$\text{the r.h.s of (30) with } (\mu = \nu) \text{ is in} \quad [\psi^{pp}_{min}(\phi,\dot{\phi},\xi), \psi^{pp}_{max}(\phi,\dot{\phi},\xi)] \tag{32}$$

- $\psi^{pp}_{min}(\phi,\dot\phi,\xi)$ and $\psi^{pp}_{max}(\phi,\dot\phi,\xi)$ are the variable lower and upper bound on $\ddot\psi$ in (22) that are compatible with the saturation constraint on $u_1$ in (24), namely

$$\psi^{pp}_{min}(\phi,\dot\phi,\xi) := \min_{\tau\in\{-u_1^{max},+u_1^{max}\}}\Big[\tau/J_r - 2\dot\phi\xi\cos^2\phi\Big]/(1-\frac{J_r}{ml^2}\sin^2\phi) \quad (33)$$

$$\psi^{pp}_{max}(\phi,\dot\phi,\xi) := \max_{\tau\in\{-u_1^{max},+u_1^{max}\}}\Big[\tau/J_r - 2\dot\phi\xi\cos^2\phi\Big]/(1-\frac{J_r}{ml^2}\sin^2\phi) \quad (34)$$

In order to understand how (28) (with $i=1$) asymptotically stabilizes $\xi$ while meeting the saturation constraint, one may analyse what happens when $\phi$ converges to 0 under the action of $u_2$. Indeed, this asymptotically leads to an admissible region for $\ddot\psi$ that contains an open neighborhood of 0 [see equations (33)-(34)]. This with (22) in which $\dot\phi = 0$ is injected clearly shows that implementing (28) yields an asymptotically stable behavior for $\xi$.

As for the design of $u_1|_{\text{mode }2}$, it is based on a sliding mode like approach in which the manifold $S = \xi - \xi_r$ is stabilized where $\xi_r$ is the control that stabilizes $\theta$ around its desired value $\theta^f$ [see equation (31)]. Since $\xi_r$ asymptotically tends to 0 with the error $\theta-\theta^f$, $\xi$ does the same. Again, variable adaptive gain $\mu$ [see equation (32)] is used in order to meet the saturation constraints on $u_1$. To end this presentation of the feedback laws $K_1$ and $K_2$, it is worth noting that the constants 0.01, 10 and 0.1 that appear in equations (27), (29), (30) and (31) are used in order to avoid very high gains near the desired targets and to obtain compatible response times in the back-stepping design approach. They might have been left as design parameters. The choice fixed here aims to avoid having too many parameters to tune.

### 4.2 Definition of the parameterized sequences $(\boldsymbol{\eta}^{\mathbf{k}}(\mathbf{p},\boldsymbol{\eta}^{\mathbf{0}}))_{\mathbf{k}\geq \mathbf{0}}$

Recall that for the snakeboard example, $\eta=\theta$. Consider the following parameterized trajectory

$$\Theta(t,p,\theta^0) := 2p_1\pi + (\theta^0 - 2p_1\pi)e^{-\lambda_\theta t} + p_2 e^{-\lambda_\theta t}\Big(1-e^{-\lambda_\theta t}\Big) \quad ; \quad \lambda_\theta > 0 \quad (35)$$

where $p=(p_1,p_2)$ belongs to the compact subset $\mathbb{P}\subset\mathbb{R}^2$ defined by

$$\mathbb{P} := \{-1,0,+1\}\times[-p_2^{max},+p_2^{max}] \quad ; \quad p_2^{max}>0 \quad (36)$$

Note that for all $p\in\mathbb{P}$, $\Theta(0,p,\theta^0)=\theta^0$ while $\lim_{t\to\infty}\Theta(t,p,\theta^0) = 0\ (mod\ 2\pi)$. Note also that the use of $p_2\neq 0$ and $p_1 = 0$ enables non constant trajectories with identical boundary conditions $\theta^0=\theta^f$ to be generated. This is crucial to obtain "*good*" solutions in some singular situations like for instance the one given by $\bar x^0 = 0$, $0\neq \bar y^0\approx 0$ and $\theta^0 = 0$. Indeed, without the parameter $p_2$, whatever small is $\bar y^0\neq 0$, an entire rotation would be necessary to steer the snakeboard to the desired position.

Given an initial value $\theta^0$ and some parameter vector $p\in\mathbb{P}$, the generation of the parameterized trajectory is done using the following three steps :

1. Choose some sampling period $\delta>0$, take an integer $q\gg \dfrac{1}{\delta\lambda_\theta}$. Generate the sequence
$$(\theta^k(p,\theta^0))_{k=0}^q \quad \text{where} \quad \theta^k(p,\theta^0) := \Theta(j\delta,p,\theta^0)$$

2. Solve the corresponding LP problem (12) to get the optimal sequence
$$\hat{v}(p,\bar{x}^0,\bar{y}^0,\theta^0)$$
Remove all vanishing (or too small) components, if any, by introducing intermediate terms in the sequence $\theta^k(p,\theta^0)$ as explained above (In fact, this has never been necessary in our experimentation).
3. Reduce the size of the resulting sequence $\theta^k(p,\theta^0)$ by keeping only the last value of all sequences corresponding to the same value of $\hat{v}(p,\bar{x}^0,\bar{y}^0,\theta^0)$. As an example, if the sequence $\hat{v}(p,\bar{x}^0,\bar{y}^0,\theta^0)$ takes the following form :
$$\hat{v}(p,\bar{x}^0,\bar{y}^0,\theta^0) = \begin{pmatrix} v_1 & v_1 & v_1 & v_2 & v_2 & v_3 & v_3 & v_3 \end{pmatrix}^T$$
then the reduced sequence $\theta^k(p,\theta^0)$ which is finally retained is the following one
$$\begin{pmatrix} \theta^3(p,\theta^0) & \theta^5(p,\theta^0) & \theta^8(p,\theta^0) \end{pmatrix}$$
this enables useless waste of time asking successive values of $\theta^k(p,\theta^0)$ to be successively approached with almost zero-velocity with the same value of $\phi$.

### 4.3 Checking the remaining assumptions

The rank condition (6) is obviously satisfied provided that the sampling period $\delta > 0$ invoked in section 4.2 is taken sufficiently small thanks to the properties of the trigonometric functions. The same can be said about the condition expressed in assumption 1.4. The translatability property naturally follows from the properties of the exponential functions used in the definition of the parameterized trajectory (35). Finally, the conditions of assumption 1.5 are clearly satisfied with $\mathcal{D} = ]-\pi/2, +\pi/2[$.

## 5 Simulation results

In this section, some simulations are proposed in order to show the efficiency of the proposed feedback algorithm. The numerical values used in the simulations as well as a recall on where each of them appears in the preceding sections is depicted in table 1. The proposed simulations aim to underline the following features

| parameter | appearing in | value | parameter | appearing in | value |
|---|---|---|---|---|---|
| $l$ | (21) | 0.5 | $J_r$ | (22) | 0.72 |
| $ml^2$ | (28) | 0.24 | $J_w$ | (23) | 0.13 |
| $\lambda_\theta$ | (35) | 0.1 | $v_{max}$ | (12) | 2 |
| $\varepsilon$ | figure 1 | 0.01 | $p_2^{max}$ | (36) | 10 |

**Table 1.** Numerical values of the system's and the controller's parameters

1. The ability to explicitly handle the saturation constraints on the control input. This may be observed by comparing figures 4 and 5 where the same initial conditions are simulated for two different saturation levels, namely $u_1^{max} = u_2^{max} = 8$ (figure 4) and $u_1^{max} = u_2^{max} = 4$ (figure 5).

2. The ability to realize rather "*economic*" trajectories when starting from some almost singular situations (like the ones shown on figures 6 and 7) avoiding whole rotations to be used.
3. Finally, figures 8 and 9 show the behavior of the closed loop under the nominal model (figure 8) and under model uncertainties (figure 9). The uncertainties are introduced in equations (22) and (23) as follows

$$\dot{\xi} = \Big(-\frac{J_r}{2ml^2}(1+\delta_1)\ddot{\psi} + \dot{\phi}\xi\Big)\cdot\tan\phi + \delta_2 \tag{37}$$

$$u_2 = 2J_w(1+\delta_3)\ddot{\phi} \quad ; \quad \delta_1 = -0.1 \quad ; \quad \delta_2 = 0.05 \quad ; \quad \delta_3 = -0.1 \tag{38}$$

   Namely, $\delta_1$ and $\delta_3$ stand for relative error on the values of the physical parameters while $\delta_2$ stands for persistant external disturbance such as wind related drift term.

## 6 Conclusion & Discussion

It has been shown that by exploiting the particular structure of a class of nonholonomic system, it is possible to derive an efficient steering procedure that can be used in a receding-horizon scheme to yield a stabilizing state feedback. By doing so, a complex dynamic problem is transformed into a rather simple discrete problem that can be solved by linear programming tools. The solution is successfully applied to the snakeboard constrained stabilization problem.

The main drawback of the proposed approach is the constraint of almost stopping motion in-between each of the two control modes being used. This constraint can be practically avoided by using a state dependent switching parameter $\varepsilon$, namely $\varepsilon(x)$ that would be large when $x$ is far from the desired state and small in its neighborhood.

## A appendix

**Lemma 1.** *Let $A \in \mathbb{R}^n$ be a regular matrix. Define a compact subset $\mathcal{C} \subset \mathbb{R}^n$ that contains a neighborhood of the origin. Let $\mathcal{C}_p \subset \mathcal{C}$ be the subset defined by*

$$\mathcal{C}_p := \Big\{ v \in \mathcal{C} \quad | \quad v_i \neq 0 \quad \forall i \in \{1, \dots, n\} \Big\} \tag{39}$$

*then there always exist $v \in \mathcal{C}_p$ such that $Av \in \mathcal{C}_p$* ♠

**Proof** Let $\mathcal{O} \subset \mathcal{C}$ be a sufficiently small neighborhood of the origin such that $A^{-1}\mathcal{O}$ is also a neighborhood of the origin that is contained in $\mathcal{C}$. The set $A^{-1}(\mathcal{O} \cap \mathcal{C}_p)$ is clearly an open neighborhood of the origin that contains an element in $\mathcal{C}_p$. Let $v$ be such an element, $v$ clearly satisfies the requirements of the lemma. □

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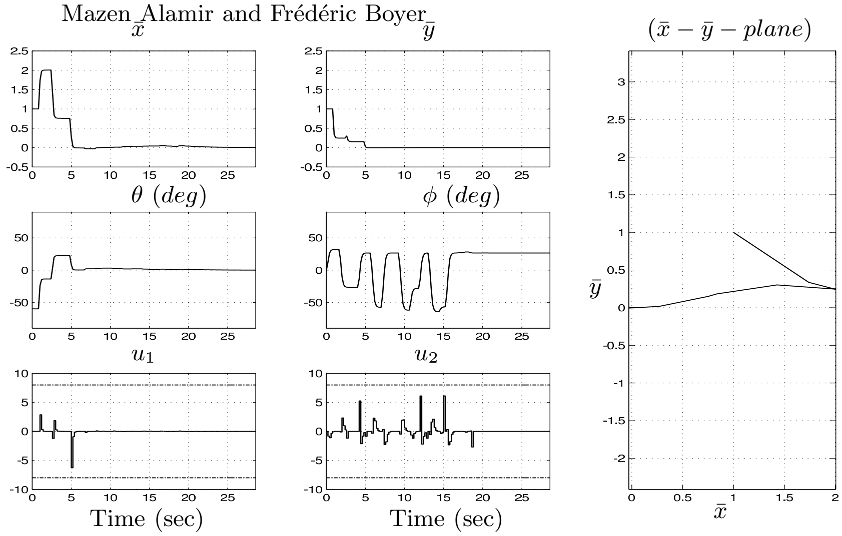


**Fig. 4.** Closed loop behavior. Initial condition $\bar{x}(0) = \bar{y}(0) = 1$, $\theta(0) = -\pi/4$ and $\xi(0) = 0$. Saturation levels $u_1^{max} = u_2^{max} = 8$. This figure is to be compared with figure 5 in order to appreciate the saturation constraints handling.

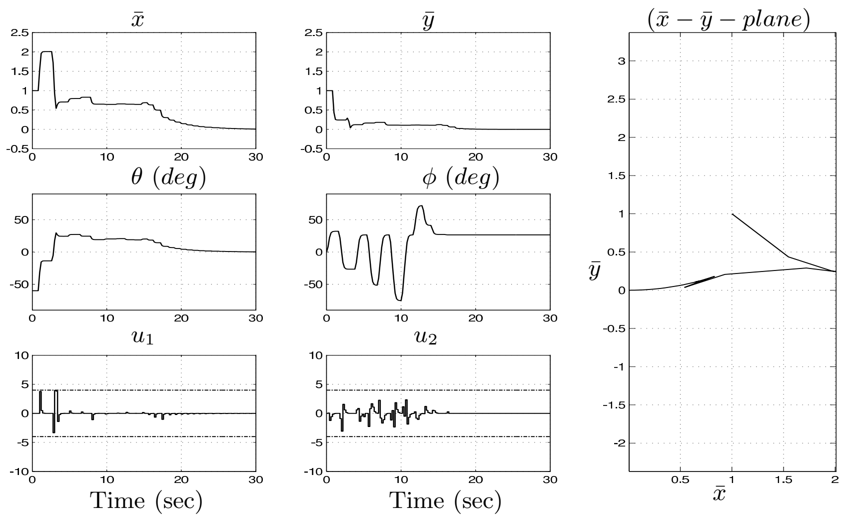


**Fig. 5.** Closed loop behavior. Initial condition $\bar{x}(0) = \bar{y}(0) = 1$, $\theta(0) = -\pi/4$ and $\xi(0) = 0$. Saturation levels $u_1^{max} = u_2^{max} = 4$. This figure is to be compared with figure 4 in order to appreciate the saturation constraints handling.

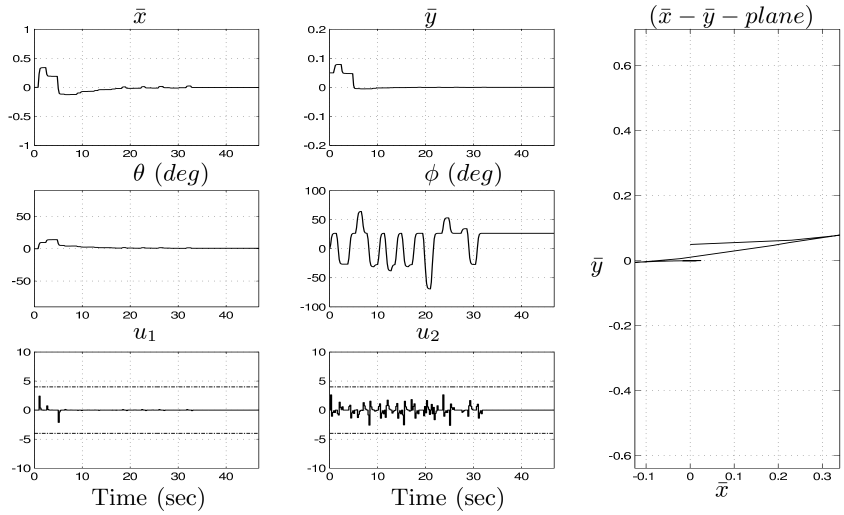


**Fig. 6.** Closed loop behavior. Initial condition $\bar{x}(0) = 0$, $\bar{y}(0) = 0.05$, $\theta(0) = 0$ and $\xi(0) = 0$. Saturation levels $u_1^{max} = u_2^{max} = 4$. Note how the parametrization of the trajectories avoids the need for a whole rotation even in this rather singular situation.

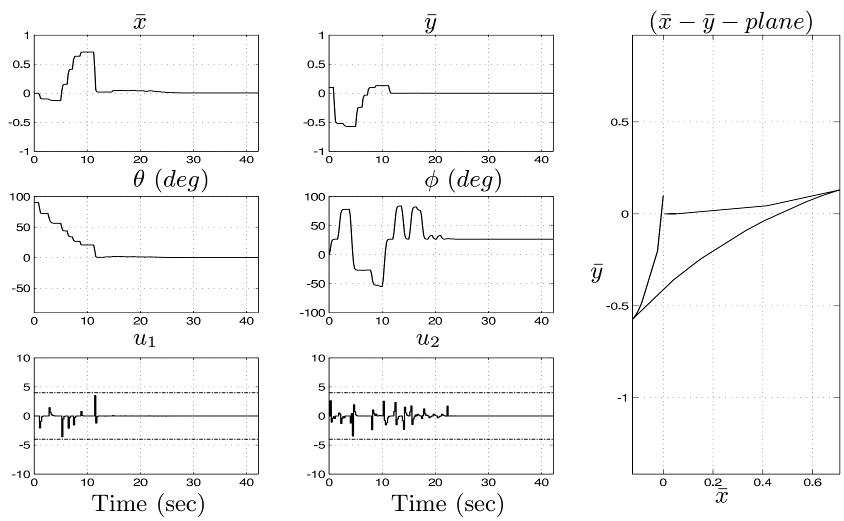


**Fig. 7.** Closed loop behavior. Initial condition $\bar{x}(0) = 0$, $\bar{y}(0) = 0.1$, $\theta(0) = \pi/2$ and $\xi(0) = 0$. Saturation levels $u_1^{max} = u_2^{max} = 4$.

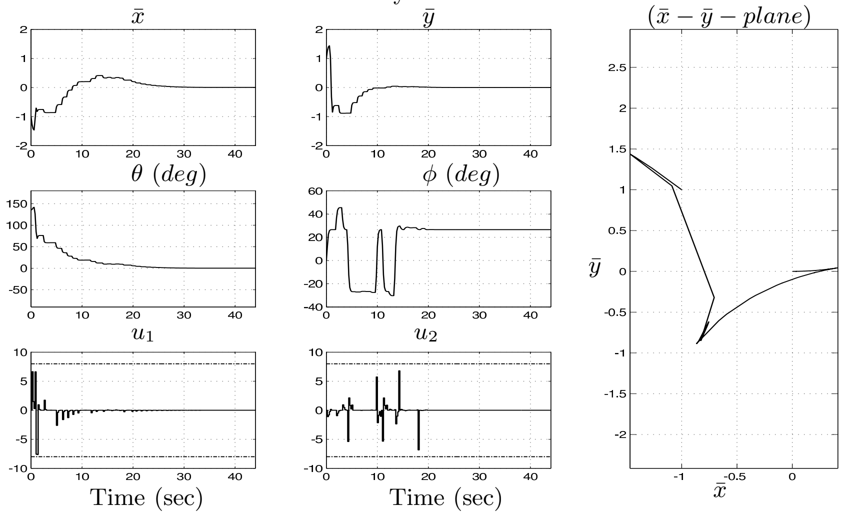


**Fig. 8.** Closed loop behavior. Initial condition $\bar{x}(0) = -1$, $\bar{y}(0) = 1$, $\theta(0) = 3\pi/4$ and $\xi(0) = 1$. Saturation levels $u_1^{max} = u_2^{max} = 8$. This simulation is done under nominal model without uncertainties. The result is to be compared to that of figure 9 where model uncertainties are introduced.

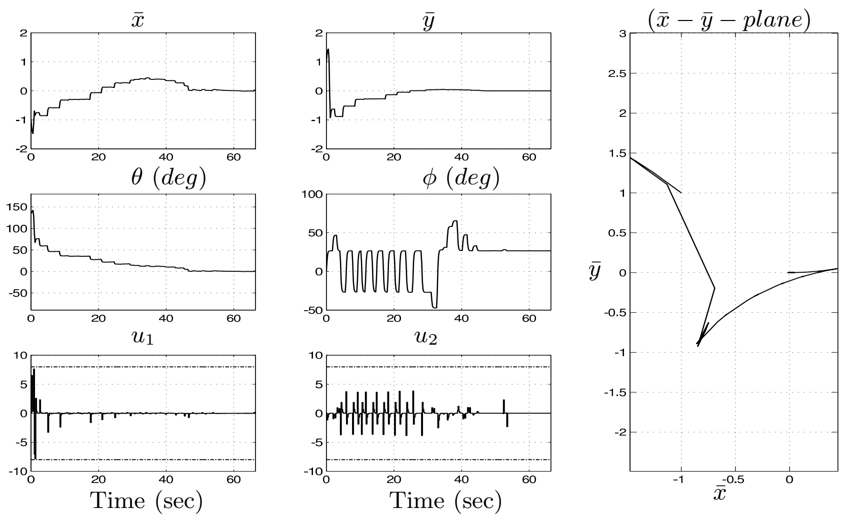


**Fig. 9.** Closed loop behavior. Initial condition $\bar{x}(0) = -1$, $\bar{y}(0) = 1$, $\theta(0) = 3\pi/4$ and $\xi(0) = 1$. Saturation levels $u_1^{max} = u_2^{max} = 8$. The uncertainties given by (37)-(38) are used to test the robustness of the proposed feedback algorithm.