Remote Output Stabilization via Communication Networks with a Distributed Control Law

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Abstract

In this paper we investigate the problem of remote stabilization via communication networks involving some time-varying delays of known average dynamics. This problem arises when the control law is remotely implemented and leads to the problem of stabilizing an open-loop unstable system with time-varying delay. We use a time-varying horizon predictor to design a stabilizing control law that sets the poles of the closed-loop system. The computation of the horizon of the predictor is investigated and the proposed control law takes into account the average delay dynamics explicitly. The resulting closed loop system robustness with respect to some uncertainties on the delay model is also considered. Simulation results are finally presented.

Index Terms

Networked control systems, stabilization with time-varying delays, observer-based control, state predictor.

I. Introduction

The networked control systems constitute a new class of control systems including specific problems such as delays, loss of information and data process. The problem studied in this paper concerns the remote stabilization of unstable open-loop systems. The sensor, actuator and system are assumed to be remotely commissioned by a controller that interchanges measurements

This work was partially supported by the NECS-CNRS projet; details on the NECS project as well as cited references from the authors can be fund at *www.lag.ensieg.inpg.fr/canudas/*.

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and control signals through a *lossless* communication network (all lost packets are re-emitted). We assume that this communication network has its own dynamics, and that a model for the average induced time-delay is available. As an example, such a model can be derived for local networks where the transfer protocol (TP) is set by the users and where a router (which can possibly inform the emitters of the instantaneous queue length) manages the packets [1], [2], [3]. Another illustrative example is proposed in [4], where the measurement of the round trip time is directly used to estimate the average value of the delay with a simple but effective algorithm.

Teleoperation of open-loop unstable systems with time-varying delays has been scarcely studied yet, which motivates the fact we don't make any assumption on the stability of the remotely controlled system. Airplane drone and tele-operated vehicles are examples of open-loop unstable and remotely controlled systems.

The control approach developed in this paper is based on the design of a state predictor and results in a "pole-placement" on the closed-loop system. Adequate placement may be ensured by classical linear methods such as H^2 or H^∞ design via LMI. The state predictor is used in [5] to achieve a finite spectrum assignment on systems with delayed output or state. The delayed input case is considered in [6], where some stabilization issues are detailed. The previous works are generalized in [7] with the concept of system reduction (infinite to finite spectrum assignment). The problem of time-varying delays is studied specifically in [8], where a precise condition on the time-varying predictor's horizon is established. This last predictor is included in a H^∞ control scheme in [9]. Based on some previous results presented in [10] and [11] (where the proposed control law was implemented in an experimental setup), the aim of our work is to detail the computation of the time-varying predictor's horizon and include the time-delay's model directly into the control scheme, with the constraints obtained from the stability analysis of the resulting closed-loop system. Another important issue considered in this paper is the robustness of the state predictor with respect to some uncertainties on the delay model. We then allow for the explicit use of some established network models (TP dynamics) in the control design.

This paper is organized as follows. The control problem considered is formulated as the problem of stabilizing a time-delay system with a state predictor which has a time-varying horizon in section II. The computation of the horizon and the *explicit* use of the average network dynamics is investigated in section III. The robustness of the resulting control setup with respect to some uncertainties on the network model is finally presented in section IV, along with a simulation example.

II. Problem Formulation

Before dealing with a particular transmission protocol dynamics, we aim at exploring how the control design can be elaborated for a system where the transmission delay is given by an autonomous stable system. More precisely, we consider systems of the form:

$$\dot{x}(t) = Ax(t) + Bu(t - \tau(t)), \quad x(0) = x_0 \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

$$\dot{z}(t) = f(z(t), u_d(t)), \quad z(0) = z_0 \tag{3}$$

$$\tau(t) = h(z(t), u_d(t)) \tag{4}$$

where $x \in R^n$ is the internal state, $u \in R$ is the control input, $y \in R^m$ is the system output, and A, B, C are matrices of appropriate dimensions. The pairs (A, B) and (A, C) are assumed to be controllable and observable, respectively, but no assumption is made on the stability of A. The signal $u_d(t)$ and the functions $f(\cdot)$ and $h(\cdot)$ are assumed to be some known continuous functions in this nominal case. These hypothesis will be relaxed later on the paper (section IV). Equation set (3)-(4) describes the internal delay dynamics representing the transmission channel model. We assume that all solutions of model (3)-(4), have the following properties for all $t \geq 0$

$$\tau_{max} \geq \tau(t) \geq 0 \tag{5}$$

$$1 - \nu \geq \dot{\tau}(t) \tag{6}$$

where $\tau_{max} \geq 0$ is an upper bound of the time-variation of $\tau(t)$ and $1 > \nu > 0$ is an arbitrarily small constant determined by the delay dynamics. These two conditions on the delay model are a direct consequence of the fact that we consider reliable transmission networks. To understand this, first note that the time-delay considered is experienced by the transmitted signal and may be different from the delay measured on the network. From this point of view, a $\dot{\tau} = 1$ means that the signal considered is blocked in the communication link indefinitely since the delay (latency) grows as fast as the current time t, which contradicts the lossless data property.

The control setup is presented on Figure 1(a). This specific location of the delay, between the control setup and the system, is motivated by the fact that most of the destabilizing effect and

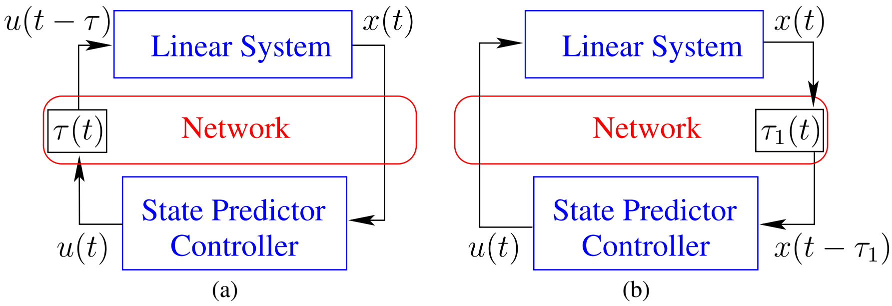

Fig. 1. Time-delay on the actuator (a) and measurement (b) signals.

technical difficulties to solve the problem come from this delay location. Indeed, if we consider an induced delay $\tau_1(t)$ located between the system and the control setup, as in Figure 1(b), then we can set the control law

$$u(t) = -K\left[e^{A\tau_1}x(t-\tau_1(t)) + e^{At}\int_{t-\tau_1}^{t} e^{-A\theta}Bu(\theta)d\theta\right] = -Kx(t)$$

where $\tau_1(t)$ is estimated or directly measured. Keeping track of the control input during the time $[t-\tau_1,\, t]$, the resulting closed-loop system has the dynamics

$$\dot{x}(t) = (A-BK)x(t)$$

and the remote stabilisation problem reduces to a traditional pole placement problem. An error in the predictor computation only introduces a consideration on the robustness with respect to some disturbances on the input signal. A setup with two delays is studied in an observer-based control scheme in [10] but will not be presented here.

A. Control design

Due to the inherent time-variation of the delay considered here, it is not possible to design a controller that imposes an invariant closed-loop spectrum. Instead, under certain weak conditions, we are able to set the eigenvalues of a *time-varying shifted* system, or equivalently we transform the time-invariant delayed unstable open-loop system, into a stable time-varying linear system. The control design proposed here is similar to the one used in [8] in an adaptive control context. The system transformation is done by replacing the current time t by the shifted time coordinate $t+\delta(t)$ in (1), which results in

$$x'(t+\delta(t)) = Ax(t+\delta(t)) + Bu(t+\delta(t)-\tau(t+\delta(t))), \tag{7}$$

where $x'(\cdot)$ is the derivative of $x(\cdot)$ with respect to its argument and $\delta(t)$ is a bounded and positive time-depending function. Defining $\delta(t)$ as

$$\delta(t) = \tau(t + \delta(t)) \tag{8}$$

and considering first the problem of state feedback stabilization, the eigenvalues of the time-varying shifted system (7) are set with the control input

$$\begin{aligned} x(t+\delta) &= e^{A\delta}\left[x(t) + e^{At}\int_t^{t+\delta} e^{-A\theta}Bu(\theta-\tau(\theta))d\theta \right] \\ u(t) &= -Kx(t+\delta(t)). \end{aligned} \tag{9}$$

The resulting closed-loop equation is then

$$x'(t+\delta(t)) = (A - BK)x(t+\delta(t)) = A_{cl}x(t+\delta(t)) \tag{10}$$

where A_{cl} is the closed loop state matrix, that can be made Hurwitz by the controllability hypothesis on the (A, B) pair.

B. Stability analysis

The stability analysis of the time-varying system (10) and the resulting constraints on the dynamics of $\delta(t)$ is detailed in the following Lemma, which proof is given in the appendix.

Lemma 2.1: Assume that $\exists \delta(t)$ satisfying (8), such that the control law (9) applied to system (7) leads to the closed-loop form (10). Then if the following conditions hold:

$i)$ All the real parts of the eigenvalues of A_{cl} are in the open left hand side of the complex plane,

$ii)$ $\infty > \delta_M \geq \delta(t) \geq 0$,

$iii)$ $\infty > \rho > \dot{\delta}(t) > -1$ with ρ an arbitrarily large positive constant.

then, $\lim_{t\to\infty} ||x(t+\delta(t))|| = 0 \quad \forall\ t+\delta(t) \geq \delta_0$ with $\delta_0 = \delta(0)$ and for all bounded values of $x(\delta_0)$. Furthermore, the state $x(t+\delta(t))$ is exponentially stable.

The stability result of the pervious lemma is applied to the system considered thanks to the following proposition.

Proposition 2.1: The hypotheses (ii) and (iii) of Lemma 2.1 are always satisfied for the delay models defined by (3)-(4) and satisfying the conditions (5)-(6).

Proof: Hypothesis (ii) is clearly satisfied from the definition of $\delta(t)$ and (5) while (iii) is obtained from (6). More precisely, taking the time-derivative of (8) and from the fact that $\dot{\tau}(t) \neq 1\, \forall t$, we can write

$$\dot{\delta}(t) = \frac{d\tau(\zeta)/d\zeta}{1 - d\tau(\zeta)/d\zeta}$$

Hypothesis (iii) is then satisfied if

$$-1 < \frac{d\tau(\zeta)/d\zeta}{1 - d\tau(\zeta)/d\zeta} < \rho.$$

The left part of this inequality clearly always holds since

$$d\tau(\zeta)/d\zeta - 1 < d\tau(\zeta)/d\zeta \Leftrightarrow -1 < \frac{d\tau(\zeta)/d\zeta}{1 - d\tau(\zeta)/d\zeta}$$

and the right part is also satisfied since (6) implies

$$\frac{1}{\nu} \geq \frac{1}{1 - d\tau(\zeta)/d\zeta} \quad and \quad \frac{d\tau(\zeta)/d\zeta}{1 - d\tau(\zeta)/d\zeta} < \frac{1 - \nu}{\nu}$$

Choosing $\rho = \dfrac{1-\nu}{\nu}$ finally ensures that ρ is finite, from the properties of ν. ■

We can finally conclude on the stability of the closed loop system with the following corollary.

Corollary 2.1: The control law (9) applied to the system (1)-(4), where the delay satisfies (5)-(6), has a bounded solution and the system trajectories exponentially decrease to zero.

III. Computation of $\delta(t)$ and use of the time-delay model

The computation of the control law implies to continuously solve (8) for $\delta(t)$ and to keep a history of the past control inputs during a time-interval $[t - \tau(t), t]$. The existence of a solution to this equation implies that $\tau(\cdot)$ satisfies (5)-(6). It is solved analytically (for specific delay models) or numerically (time consuming) in [11]. A more convenient and efficient way to compute $\delta(t)$ is to use directly the delay dynamics. This is achieved by first defining the function

$$s(t) = \hat{\delta}(t) - \tau(t + \hat{\delta}(t)) \tag{11}$$

where $\hat{\delta}(t)$ is the computed estimate of $\delta(t)$. The idea is to find a variation law for $\hat{\delta}$ such that the manifold $s(t) = 0$ is rendered attractive and invariant, consequently ensuring that $\hat{\delta}$ converges asymptotically to δ. In order to prevent for the numerical instabilities, the dynamics of $s(t)$ is defined as

$$\dot{s}(t) + \lambda s(t) = 0 \tag{12}$$

where λ is a positive constant. Deriving (11) with respect to time and substituting $\dot{s}$ in (12), we obtain

$$\dot{\hat{\delta}} - \tau'(\hat{\zeta})(1 + \dot{\hat{\delta}}) + \lambda(\hat{\delta} - \tau(\hat{\zeta})) = 0 \tag{13}$$

where $\hat{\zeta}(t) = t + \hat{\delta}(t)$ and $\tau'(\cdot)$ is the derivative of $\tau(\cdot)$ with respect to its argument. From the previous equation, (12) is satisfied if $\tau'(\cdot) \neq 1$ and the variation law $\dot{\hat{\delta}}(t)$ is set with

$$\dot{\hat{\delta}}(t) = -\frac{\lambda\hat{\delta}}{1 - \tau'(\hat{\zeta})} + \frac{\tau'(\hat{\zeta}) + \lambda\tau(\hat{\zeta})}{1 - \tau'(\hat{\zeta})} \tag{14}$$

This *explicit* expression for the dynamics of $\hat{\delta}(t)$ then ensures that the approximate $\hat{\delta}(t)$ converges to the desired value $\delta(t)$, and that the function $s(t)$ exponentially converges to zero. The convergence speed can be set arbitrarily fast by choosing a small ϵ, and we directly use the delay dynamics ($\tau(\hat{\zeta})$ and $\tau'(\hat{\zeta})$ are given by (3)-(4)). To illustrate the computation of $\dot{\hat{\delta}}$, consider the case where $\tau(t) = z(t)$: (14) is then set using $\tau(\hat{\zeta}) = z(\hat{\zeta})$ and $\tau'(\hat{\zeta}) = z'(\hat{\zeta}) = f(z(\hat{\zeta}), u_d(\hat{\zeta}))$. The influence of the dynamics of $s(t)$ introduced in (11) on the closed-loop system is studied with the following lemma, which is a synthesis of the results presented in [12].

Lemma 3.1: Consider the closed-loop system described by

$$x'(t + \delta) = A_{cl}x(t + \delta) + BK[x(t + \delta) - x(t + \hat{\delta})], \quad x(0) = x_0 \tag{15}$$

with $\hat{\delta}$ obtained from (14). If

- $\tau(t)$ satisfies the properties (5)-(6),
- A_{cl} is a Hurwitz matrix,
- $0 < \lambda < \dfrac{1 - \nu}{2|\hat{\delta}(0) - \tau(\hat{\delta}(0))|}$,

then the trajectories of $x(t + \delta)$ are asymptotically stable.

Proof: (Outline) The previous lemma is established from the fact that the stability of the transformed system

$$\Sigma_t : x'(\zeta) = (A - BK)x(\zeta) + BKA\int_{-\epsilon_\delta}^{0} x(\zeta + \theta)d\theta - (BK)^2\int_{-2\epsilon_\delta}^{-\epsilon_\delta} x(\zeta + \theta)d\theta$$

where $\epsilon_\delta(t) = \delta(t) - \hat{\delta}(t)$ and $\zeta(t) = t + \delta(t)$, implies the stability of (15). This transformed system is obtained using the Leibniz-Newton formula

$$x'(\zeta) = (A - BK)x(\zeta) + BK\int_{-\epsilon_\delta}^{0} x'(\zeta + \theta)d\theta$$

The behaviour of Σ_t is then investigated thanks to the Lyapunov-Krasovskii functional [13]

$$\begin{aligned} V(x(\zeta)) &= x(\zeta)^T P x(\zeta) + \frac{1}{1-\bar{\epsilon}_\delta}\int_{-\epsilon_\delta}^{0}\left[\int_{\zeta+\theta}^{\zeta} x(\mu)^T S x(\mu) d\mu\right] d\theta \\ &+ \frac{\alpha}{1-2\bar{\epsilon}_\delta}\int_{-2\epsilon_\delta}^{-\epsilon_\delta}\left[\int_{\zeta+\theta}^{\zeta} x(\mu)^T S x(\mu) d\mu\right] d\theta \end{aligned}$$

with P, S some positive definite matrices, $\bar{\epsilon}_\delta \doteq \sup_t \dot{\epsilon}_\delta(t)$ and

$$0 < \alpha < \frac{1-2\bar{\epsilon}_\delta}{\bar{\epsilon}_\delta}$$

Taking the time-derivative of this functional along the system trajectories of (14)-(15) ensures, if the hypotheses of the lemma are satisfied, the stability of the system considered. ■

The previous lemma is now applied to the proposed control scheme in the following theorem.

Theorem 3.1: Consider the system (1) with (A, B) a controllable pair. Assume that the delay dynamics (3)-(4) is such that (5)-(6) hold for all t, then the feedback control law (9) based on the estimated predictor's horizon $\hat{\delta}(t)$ which dynamics are described by (14) with

$$\begin{aligned} \tau'(\hat{\zeta}) &= \frac{dh}{d\hat{\zeta}}(z(\hat{\zeta}), u_d(\hat{\zeta})) \\ \frac{dz}{d\hat{\zeta}}(\hat{\zeta}) &= f(z(\hat{\zeta}), u_d(\hat{\zeta})), \quad z(0) = z_0 \end{aligned}$$

and λ satisfying the conditions stated in lemma 3.1, ensures that the trajectories of $x(t)$ decrease asymptotically to zero.

Proof: First note that the time-shifted system

$$x'(t+\delta) = Ax(t+\delta) + Bu(t)$$

with $u(t) = -Kx(t+\hat{\delta})$ writes as (15) by adding and subtracting $BKx(t+\delta)$ to the previous dynamic equation. Thanks to Lemma 3.1 and conditions (5)-(6), the proposed control law then allows for a pole placement on the time-shifted system described by the state $x(t+\delta)$ and A_{cl} in (10) is made Hurwitz with a proper choice of K. Therefore, the time-shifted state converges asymptotically with the proposed control law. Finally, the stability of $x(t)$ is deduced from the fact that the system (1)-(2) is linear and its states cannot diverge in finite time. ■

IV. Robustness Analysis

The aim of this section is to study the robustness of the system (1)-(4) stabilized by the state feedback (9) with respect to some delay uncertainties. These uncertainties are due to the difference that may exist between the delay model (3)-(4) and the true delay induced by the communication channel.

A. Problem formulation

In order to study the robustness of the control setup with respect to delay uncertainties, we investigate their influence on the dynamics of the closed-loop system. The dynamics of the estimated delay $\hat{\tau}$ is obtained from

$$\dot{\hat{z}}(t) = f_e(\hat{z}(t), u_{de}(t)), \quad \hat{z}(0) = \hat{z}_0 \tag{16}$$

$$\hat{\tau}(t) = h_e(\hat{z}(t), u_{de}(t)) \tag{17}$$

where $f_e(\cdot)$ and $h_e(\cdot)$ are some continuous functions, $\hat{z}$ is the internal state of the model and u_{de} is an exogenous input to this model, possibly including some network measurements. The estimated delay satisfies the conditions

$$\begin{array}{c} \hat{\tau}_{max} \geq \hat{\tau}(t) \geq 0 \\ \sup \dot{\hat{\tau}}(t) = \hat{\nu} < 1 \end{array} \tag{18}$$

Considering that such a model exists and is compared to the actual network induced delay with the error parameters ϵ_M and $\bar{\epsilon}_M$ defined as

$$\epsilon(t) \doteq \tau(t) - \hat{\tau}(t) \tag{19}$$

$$\{\epsilon_M, \bar{\epsilon}_M\} \doteq \sup_t \{\epsilon(t), \dot{\epsilon}(t)\} \tag{20}$$

the aim of this section is to determine if, for a chosen feedback gain K, the closed-loop system remains stable when $\{\epsilon_M, \bar{\epsilon}_M\} \neq \{0, 0\}$. The predicted state feedback is computed from the delay model and the resulting closed-loop system writes as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t - \tau(t)) \\ u(t) &= -Ke^{A\hat{\delta}(t)} \left[x(t) + e^{At} \int_t^{t+\hat{\delta}(t)} e^{-A\theta} Bu(\theta - \hat{\tau}(\theta)) d\theta \right] \\ \hat{\delta}(t) &= \hat{\tau}(t + \hat{\delta}(t)) \end{aligned}$$

where $\hat{\delta}(t)$ is the prediction horizon computed from (16)-(17) ($\hat{\tau} \neq \tau \Rightarrow \hat{\delta} \neq \delta$).

The controller output $u(t)$ can be expressed, equivalently, as

$$u(t) = -Kx(t+\hat{\delta}(t)) + \Delta_u(t)$$

where

$$\Delta_u(t) \doteq -Ke^{A(t+\hat{\delta}(t))} \int_t^{t+\hat{\delta}(t)} e^{-A\theta} B[u(\theta - \hat{\tau}(\theta)) - u(\theta - \tau(\theta))]d\theta$$

The resulting closed-loop system is then defined by the functional differential equation

$$x'(t+\delta(t)) = Ax(t+\delta(t)) - BKx(t+\hat{\delta}(t)) + B\Delta_u(t) \tag{21}$$

While a direct Lyapunov-Krasovskii analysis (similar to the one used in the previous section) of this problem is very conservative [14], some more interesting results can be obtained by neglecting the effect of Δ_u in the previous dynamics. This is motivated, intuitively, as follows. From the fact that

- Δ_u is proportional to the difference $u(\theta - \hat{\tau}(\theta)) - u(\theta - \tau(\theta))$ and is bounded since there is no singularity in the system and the integration is carried on a finite-time horizon,
- $|u(t)|$ is proportional to $|x(t+\hat{\delta}(t))|$,

then $|\Delta_u|$ is proportional to the distance

$$|x(\theta) - x(\theta - \tau(\theta) + \hat{\delta}(\theta - \tau(\theta)))|$$

where $\theta \in [t, t+\hat{\delta}(t)]$. If we suppose that this distance is sufficiently small to ensure the robustness of the origin stability, then the stability of (21) can be deduced from the stability of

$$x'(t+\delta(t)) = Ax(t+\delta(t)) - BKx(t+\hat{\delta}(t)) \tag{22}$$

Note that this is a qualitative result based on the vanishing perturbation theory [15]. From a physical point of view, it is equivalent to consider that the main disturbing effect of the delay estimation error acts on the fundamental dynamics of (21).

B. *Proposed solution*

We consider here the *small gain* approach for time-delay systems proposed in [16], applied to the stability analysis of

$$\begin{aligned} x'(t+\delta(t)) &= Ax(t+\delta(t)) - BKx(t+\hat{\delta}(t)) \\ &= (A-BK)x(t+\delta(t)) + BK\left(x(t+\delta(t)) - x(t+\hat{\delta}(t))\right) \end{aligned} \tag{23}$$

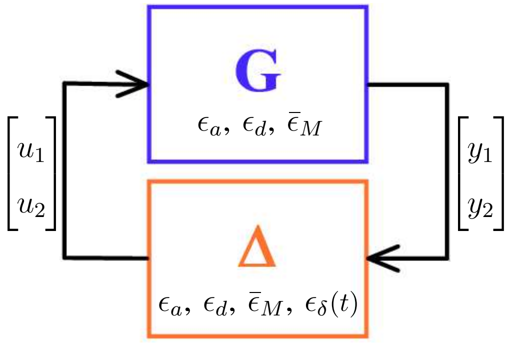

Fig. 2. Small gain formulation.

The previous equation is first written as a function of the average, constant value of the error ϵ_a (i.e. $\epsilon_a = [\max \epsilon(t) - \min \epsilon(t)]/2$) thanks to the relationship

$$x(t - \epsilon_\delta(t)) = x(t - \epsilon_a) - \int_{t-\epsilon_\delta(t)}^{t-\epsilon_a} \dot{x}(\theta) d\theta$$

where $\epsilon_\delta(t) \doteq \delta(t) - \hat{\delta}(t)$. Note that the average and maximum values of $\epsilon_\delta(t)$ are the same as those of $\epsilon(t)$. The dynamics of the resulting system is then

$$x'(\zeta) = Ax(\zeta) - BKx(\zeta - \epsilon_a) + BK \int_{\zeta-\epsilon_\delta(t)}^{\zeta-\epsilon_a} [Ax(\theta) - BKx(\theta - \epsilon_\delta(\theta))] \, d\theta$$

with $x(\theta) = \phi(\theta)$, $\theta \in [t_0 - \epsilon_M, \, t_0]$, $(t_0, \, \phi) \in \mathbb{R}^+ \times \mathcal{C}^\nu_{n, \, \epsilon_M}$. The integral term in the previous equality is considered as an uncertainty and the closed-loop system writes as

$$y_{sg} = G(u_{sg}), \quad u_{sg} = \Delta(y_{sg})$$

where $y_{sg} = [y_1 \, y_2]^T$, $u_{sg} = [u_1 \, u_2]^T$, and G and Δ are defined as

$$G \; : \; \begin{cases} x'(\zeta) &= Ax(\zeta) - BKx(\zeta - \epsilon_a) + \epsilon_d BK u_2(\zeta) \\ y_1(t) &= \dfrac{1}{\sqrt{1 - \bar{\epsilon}_M}} x(t) \\ y_2(t) &= Ax(t) - BKu_1(t) \end{cases} \tag{24}$$

$$\Delta \; : \; \begin{cases} u_1(t) &= \Delta_1 y_1(t) = \sqrt{1 - \bar{\epsilon}_M} y_1(t - \epsilon(t)) \\ u_2(t) &= \Delta_2 y_2(t) = \dfrac{1}{\epsilon_d} \displaystyle\int_{t-\epsilon_\delta(t)}^{t-\epsilon_a} y_2(\theta) d\theta \end{cases} \tag{25}$$

where $\epsilon_d \doteq max\{\epsilon_M - \epsilon_a; \epsilon_a - \epsilon_m\}$ and $\epsilon_m \doteq \inf_t(\epsilon(t))$. The interconnection between G and Δ is presented in figure IV-B. Note that this specific formulation aims at separating the expressions with constant (in G) and time dependent (in Δ) values of $\epsilon_\delta(t)$. The stability of the interconnected system is obtained by showing that the gain of both subsystems G and Δ are less then one. The main advantage of this formulation is that the stability of the closed loop system is inferred from the stability of G, which is a system with a *constant* time-delay. More precisely, we first consider the following result [16]

$$\gamma_0(\Delta_k X_k) \leq 1, \text{ for all non-singular matrices } X_k \in \mathbb{R}^{n\times n},\ k = 1, 2,$$

where $\gamma_0(H)$ is the gain of the system considered and H_X is the new system, defined respectively as

$$\gamma_0(H) = \inf\{\gamma|\ ||Hf||_2 \leq \gamma ||f||_2 \text{ for all } f \in L_{2+}\}, \text{ and}$$
$$H_X f = XH(X^{-1}f).$$

———— check with [Gu] for the notation... unclear ————

L_{2+} denotes the set of functions $f : \overline{\mathbb{R}}_+ \rightarrow \mathbb{R}^n$, $\overline{\mathbb{R}}$ being the closed set of square integrable reals, i.e., $\int_0^\infty ||f(t)||^2$ is well defined and finite. We can then conclude on the stability of (23) by applying the following proposition.

Proposition 4.1: [16] The input-output stability of system (23) is ensured if the scaled small gain problem

$$\gamma_0(G_X) < 1 \text{ for } X = diag(X_1\ X_2),\ X_1, X_2 \in \mathbb{R}^{n\times n} \text{ non singular}$$

has a solution, where G is described by (24).

Consequently, we have to find the sufficient conditions that the estimation error has to fulfil in order to ensure that the gain of G is bounded by one. This is done with the following proposition

Proposition 4.2: Consider the system G described by

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-r) + Eu(t) \\ y(t) &= G_0 x(t) + G_1 x(t-r) + Du(t), \end{aligned}$$

a given set of non-singular matrices $\mathcal{X}$, and $\mathcal{Z} \doteq \{X^T X | X \in \mathcal{X}\}$. There exists a $X \in \mathcal{X}$ such that $\gamma_0(G_X) < 1$ if there exists a $Z \in \mathcal{Z}$ and real matrices $P = P^T$, Q_p, S_p, $R_{pq} = R_{qp}^T$,

$p = 0, 1, \dots, N$, $q = 0, 1, \dots, N$ such that the following LMIs are satisfied:

$$\begin{pmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R} + \tilde{S} \end{pmatrix} > 0$$

$$\begin{pmatrix} \tilde{\Delta} & -\tilde{D}^s & -\tilde{D}^a \\ -\tilde{D}^{sT} & R_d + S_d & 0 \\ -\tilde{D}^{aT} & 0 & 3S_d \end{pmatrix} > 0$$

where

$$\tilde{Q} \doteq (Q_0 \quad Q_1 \quad \dots \quad Q_N), \quad \tilde{S} \doteq \frac{1}{h} diag(S_0 \quad S_1 \quad \dots \quad S_N)$$

$$\tilde{R} \doteq \begin{pmatrix} R_{00} & R_{01} & \dots \\ R_{01}^T & \ddots & \\ \vdots & & R_{NN} \end{pmatrix}, \quad h = \frac{r}{N}$$

$$\tilde{\Delta} \doteq \begin{pmatrix} \Delta_{00} & Q_N - PA_1 - G_0^T Z G_1 & -PE - G_0^T Z D \\ (*) & S_N - G_1^T Z G_1 & -G_1^T Z D \\ (*) & (*) & Z - D^T Z D \end{pmatrix}$$

$$\Delta_{00} \doteq -PA_0 - A_0^T P - Q_0 - Q_0^T - S_0 - G_0^T Z G_0$$

$$\tilde{D}^s \doteq \begin{pmatrix} D_1^s & \dots & D_N^s \end{pmatrix}$$

$$D_p^s \doteq \begin{pmatrix} \frac{h}{2} A_0^T (Q_{p-1} + Q_p) + \frac{h}{2}(R_{0,p-1} + R_{0,p}) - (Q_{p-1} - Q_p) \\ \frac{h}{2} A_1^T (Q_{p-1} + Q_p) - \frac{h}{2}(R_{N,p-1} + R_{N,p}) \\ \frac{h}{2} E^T (Q_{p-1} + Q_p) \end{pmatrix}$$

$$\tilde{D}^a \doteq \begin{pmatrix} D_1^a & \dots & D_N^a \end{pmatrix}$$

$$D_p^a \doteq \frac{h}{2} \begin{pmatrix} -A_0^T (Q_{p-1} - Q_p) - (R_{0,p-1} - R_{0,p}) \\ -A_1^T (Q_{p-1} - Q_p) + (R_{N,p-1} - R_{N,p}) \\ E^T (Q_p - Q_{p-1}) \end{pmatrix}$$

$$R_d \doteq \begin{pmatrix} R_{d11} & R_{d12} & \dots \\ R_{d12}^T & \ddots & \\ \vdots & & R_{dNN} \end{pmatrix}, \quad R_{dpq} \doteq h(R_{p-1,q-1} - R_{pq})$$

$$S_d \doteq diag\,(S_{d1} \quad S_{d2} \quad \dots \quad S_{dN}), \quad S_{dp} \doteq S_{p-1} - S_p$$

This result is applied to the system considered by using $h = \dfrac{\epsilon_a}{N}$ and

$$\begin{cases} A_0 = A, \quad A_1 = -BK, \quad E = [0_{n\times n} \;\; \epsilon_d BK] \\ G_0 = \begin{bmatrix} \dfrac{1}{\sqrt{1-\bar{\epsilon}_M}} I_n \\ A \end{bmatrix}, \quad G_1 = 0_{2n\times n}, \quad D = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} \\ -BK & 0_{n\times n} \end{bmatrix} \end{cases} \tag{26}$$

from which we can compute the maximum average estimation error on the delay, for a given maximum variation of this error.

Example 4.1: Consider the "T" shape inverted pendulum described by the dynamics

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -18.78 & 0 & 14.82 & 0 \\ 0 & 0 & 0 & 1 \\ 56.92 & 0 & -15.18 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 7.52 \\ 0 \\ -8.82 \end{bmatrix} u(t-\tau(t))$$

The controller gain K is chosen such that the poles of the matrix A_{cl} are $[-8+0.5i; -8-0.5i; -16; -32]$. The results obtained in proposition 4.2 are applied with the relationships (26) and $\epsilon_d = 2\epsilon_a$. The estimated delay average $\hat{\tau}(t)$ is based on the model proposed in [1] and depicted in figure 3(a).

This example aims at illustrating the fact that the closed-loop system remains stable if the error fits within the bounds estimated in this section. We suppose that the error and estimated delay maximum variations are the same: $\bar{\epsilon}_M = \hat{\nu} = 0.6167$, which gives $\epsilon_a = 5.9ms$. The error trial function is

$$\epsilon(t) = \epsilon_a + \epsilon_a \sin\left(\frac{\bar{\epsilon}_M}{\epsilon_a} t\right)$$

and we study the system response when the actual delay is

$$\tau(t) = \hat{\tau}(t) + \epsilon(t) \quad or \quad \tau(t) = \hat{\tau}(t) - \epsilon(t)$$

The system response to a non zero initial condition is presented in figure 3(a) in the first case and in figure 3(b) in the second case. The oscillating delay $\tau(t)$ applies to the data travelling from the control setup to the system, while the estimation $\hat{\tau}(t)$ is used to compute the predictor horizon. The time evolutions of the pendulum angle $\theta(t)$ and position $z(t)$ illustrate the sensitivity of the system to the estimation error.

This simulation result illustrates the capability of the proposed control law to stabilize the system considered when the error satisfies the conditions established in this section. Note that the closed-loop system fails to stabilize if ϵ_a is increased by $2\,ms$.

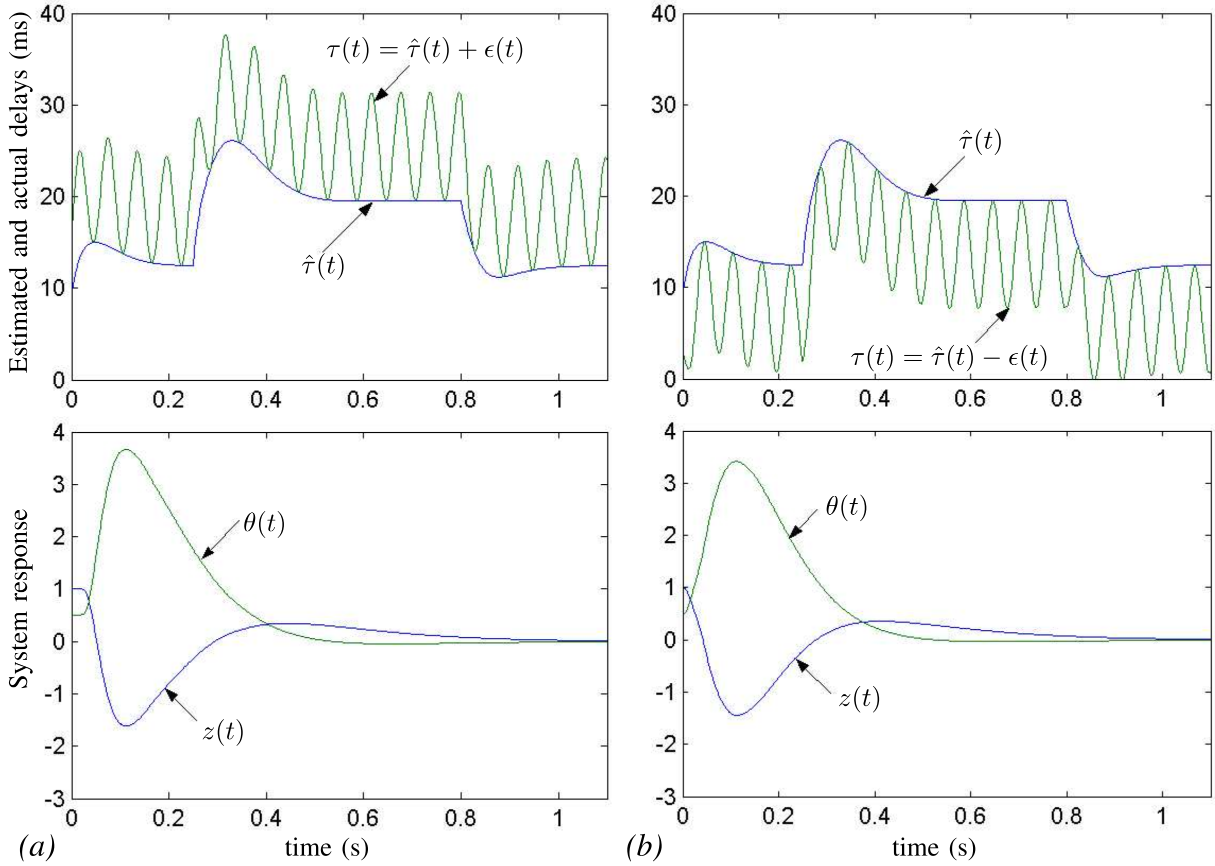

Fig. 3. Influence of the delay estimation error.

V. CONCLUSIONS

In this paper we have investigated the problem of remote stabilization via communication networks, which is formulated as the problem of stabilizing an open-loop unstable system with a time-varying delay with known dynamics. The proposed controller results in an exponentially converging closed-loop system, under weak assumptions. The controller is based on a $\delta(t)$-step ahead predictor, where $\delta(t)$ is the solution of the implicit equation $\delta - \tau(t + \delta) = 0$, which is shown to be solved if the time delay is bounded. A dynamic solution of this equation is detailed, allowing for the explicit use of the average network dynamics in the control law. The robustness of the control law with respect to time-delay uncertainties is also studied and a LMI formulation allows to compute the maximum admissible bounds on the delay estimation error. We have presented a simulation showing the capability of this controller to robustly stabilize a system when the average delay is estimated and the actual delay satisfies some computed error bounds.

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APPENDIX

PROOF OF LEMMA 2.1 (EXCERPT FROM[11])

Proof: [Lemma 2.1] Introduce $\zeta(t) = t + \delta(t)$, then (10) writes as

$$\frac{d[x(\zeta)]}{dt} = \frac{dx(\zeta)}{d\zeta}\frac{d\zeta}{dt} = (1 + \dot{\delta})A_{cl}x(\zeta)$$

Note that this equation describes a linear time-variant system in the shifted time-coordinates $\zeta(t)$. Introducing the Lyapunov function

$$V(t) = x(\zeta(t))^T P x(\zeta(t)) \tag{27}$$

we get together with the assumption (i), that there exist a positive definite matrix Q such that

$$\begin{aligned} \dot{V}(t) &= (1 + \dot{\delta}(t))x(\zeta)^T(PA_{cl} + A_{cl}^T P)x(\zeta) \\ &= -(1 + \dot{\delta}(t))x(\zeta)^T Q x(\zeta) \\ &\leq -(1 + \dot{\delta}(t))\lambda_m(Q)||x(\zeta)||^2 < 0 \end{aligned} \tag{28}$$

where the last inequality comes from the assumption (iii). Using the bounds $\lambda_M(P)||x(\zeta)||^2 \geq V(\zeta) \geq \lambda_m(P)||x(\zeta)||^2$ and integrating this inequality between 0 and t, we get

$$V(t) \leq V(0)e^{-\Psi(t)}$$

where

$$\Psi(t) \doteq \frac{\lambda_m(Q)}{\lambda_M(P)} \int_0^t (1 + \dot{\delta}(\theta))d\theta > 0$$

from which we obtain the relation

$$||x(\zeta)||^2 \leq \frac{\lambda_M(P)}{\lambda_m(P)}||x(\delta_0)||^2 e^{-\Psi(t)}, \tag{29}$$

It remains to be shown that $\Psi(t) \to \infty$ as $t \to \infty$. For this, we use the definition of $\Psi(t)$ to show that

$$\Psi(t) = \frac{\lambda_m(Q)}{\lambda_M(P)}(t + \delta(t) - \delta_0)$$

From the boundedness of $\delta(t)$, we have that $t \to \infty$ implies that $\lim_{t\to\infty} \Psi(t) = \infty$, which applied to the relation (29), concludes the proof. ∎