Asymptotic controllability implies continuous-discrete time feedback stabilizability

Nicolas Marchand[1] and Mazen Alamir[2]

[1] Laboratoire d'Automatique et de Génie des Procédés,
UCLB-Lyon 1, 43 bd du 11 Novembre 1918, 69622 Villeurbanne, France
[2] Laboratoire d'Automatique de Grenoble,
ENSIEG BP 46, 38402 Saint Martin d'Hères Cedex, France

Abstract. In this paper, the relation between asymptotic controllability and feedback stabilizability of general nonlinear systems is investigated. It is proved that asymptotic controllability implies for any strictly positive sampling period of a stabilizing feedback in a continuous-discrete time framework. The proof uses receding horizon considerations to construct a stabilizing feedback.

1 Introduction

For linear systems, it is well known that a system is (globally) asymptotically stabilizable by means of a state feedback if and only if it is (globally) asymptotically null controllable. Furthermore, the stabilizing feedback may be chosen smooth[1]. Does an analogous property exist for general nonlinear systems of the form (1) remains an open question in the nonlinear control theory framework.

$$\dot{x} = f(x, u) \tag{1}$$

Brockett et al. [1] gave the following three necessary conditions for the existence of a stabilizing $\mathcal{C}^1$ feedback:

Theorem 1 (Brockett et al. [1]). *If f is $\mathcal{C}^1$ and the feedback $u \in \mathcal{C}^1$ is such that $u(0) = 0$ and the origin is asymptotically stable for $\dot{x} = f(x, u(x))$, then:*

1. *there is no uncontrollable modes of the linearized system associated with eigenvalues with nonnegative real parts*
2. *the origin is attractive*
3. *f maps every neighbourhood of the origin onto a neighbourhood of the origin*

Applied on a linear system $\dot{x} = Ax + Bu$, it simply gives that $[A : B]$ has to be full rank which follows from the asymptotic null controllability assumption.

[1] infinitely differentiable

When only continuous feedbacks are considered, the first condition is no more necessary as shown by Kawski [6]. However, the third condition remains even if f is only continuous [14]. This last condition, often quoted in the literature as the Brockett's condition, shows that there is no hope to obtain a general relation between asymptotic controllability and feedback stabilizability if one imposes regularity assumptions on the feedback.

Remained to explore the field of discontinuous feedback. Indeed, discontinuous feedbacks arise often in many areas of control theory as well as practice. Nevertheless, it immediately yields to the difficulty: how should be defined the solution of (1) when $u(x)$ is discontinuous ? The best known theoretical tool for this is the Filippov theory [4]. Unfortunately, it was shown in [10] that it also yields to the Brockett's necessary condition. Moreover, it is proved in [3] that for affine in controls systems, the existence of a stabilizing feedback in the Filippov sense implies the existence of a non stationary continuous feedback.

In a recent paper, Clarke et al. took a slightly different approach [2]. Instead of considering the continuous time solution of system (1), which may even not be defined if no regularity is assumed, the solution used is the one of a continuous-discrete time system. With this approach, it can be established that asymptotic controllability implies a particular type of continuous-discrete time feedback stabilizability (see Th. 2). This mainly follows from a theorem established in [9] and generalized in [11], that argues, roughly speaking, that for asymptotically controllable systems, there always exists a continuous Lyapunov function V that can be decreased by means of a control. Using a regularization theorem and Rademacher's theorem, it is proved that one can find a feedback that makes decrease a sufficiently precise lipschitz local approximation of V so that V also decreases.

In this paper, the assumptions and the continuous-discrete time solutions considered are identical to Clarke's work [2]. With these assumptions, the asymptotic controllability is proved to imply the existence of a feedback that asymptotically stabilizes the system in continuous-discrete time, whatever the sampling period $T > 0$. The present result has the advantage of ensuring the asymptotic stability of the continuous-discrete time closed loop system, when only practical stability was obtained with Clarke's result. On the other hand, it is not possible (at least simply) to make the sampling period tend to the zero in order to get *generalized solutions* of the closed loop equation $\dot{x} = f(x, k(x))$ (see [13]).

The paper is organized as follows. After some preliminary definitions, the main result (that is Th. 3), is presented in Sec. 2. The last section is dedicated to its proof.

2 Problem statement and main result

2.1 Preliminary definitions

The system considered in this paper is of the form (1) where f is assumed to be continuous and locally lipschitz in x uniformly w.r.t. to u. This assumption ensures the existence and the uniqueness, for any essentially bounded control u and initial condition x_0, of a trajectory $x(.;x_0,u)$, solution of the initial valued problem $\{\dot{x} = f(x,u(t)), x(0) = x_0\}$. This will not be practically restrictive since only uniformly bounded controls will be considered in the following. The system is assumed to be globally asymptotically controllable, that is [12]:

Definition 1 (global asymptotic controlability). System (1) is said to be *globally asymptotically controllable* if there is a measurable function $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^p$ with for all $x \in \mathbb{R}^n$, $u(x,.) \in \mathcal{L}_\infty^{\mathbb{R}^p}$ such that:

1. (attractivity) $\forall x_0 \in \mathbb{R}^n, \quad \lim_{t\to\infty} x(t;x_0,u(x_0,.)) = 0$
2. (stability) $\forall R > 0$, $\exists r(R) > 0$ such that $\forall x_0 \in \mathcal{B}(r(R))$, one has $x(t;x_0,u(x_0,.)) \in \mathcal{B}(R)$ for all $t \geq 0$.

where $\mathcal{L}_\infty^{\mathbb{R}^p}$ denotes the set of functions $f : \mathbb{R} \to \mathbb{R}^p$, essentially bounded on every compact set $[a,b]$

Furthermore, to rule out the case when an infinite control is required to bring the state of the system to the origin, one assumes that:

Assumption 1. *There exists a neighbourhood of the origin $\mathcal{V}(0) \subset \mathbb{R}^n$ and a compact set $\mathcal{U} \subset \mathbb{R}^p$ such that for all $x_0 \in \mathcal{V}(0)$, there exists a function u satisfying the above definition such that $u(x_0,t) \underset{a.a.\,t}{\in} \mathcal{U}$ for almost all t.*

Some definitions that enables a proper definition of (a) solution(s) to the closed loop system are next given. Let a partition of $\mathbb{R}^+$ be defined by:

Definition 2 (partition). Every series $\pi = (t_i)_{i\in\mathbb{N}}$ of positive real numbers such that $t_0 = 0$, $\forall i,j \in \mathbb{N}, \quad t_i < t_j$ and $\lim_{i\to\infty} t_i = +\infty$ will be called a partition. Furthermore, let (when it makes sense):

- $\bar{d}(\pi) := \sup_{i\in\mathbb{N}}(t_{i+1} - t_i)$ be the **upper diameter** of π,
- $\underline{d}(\pi) := \inf_{i\in\mathbb{N}}(t_{i+1} - t_i)$ be the **lower diameter** of π.

With the above definition, one can define the notion of π-trajectory that can be seen as a continuous-discrete time solution of (1). This is an intermediate between the classical continuous time approach $\dot{x} = f(x,k(x))$ and the Euler integration giving $\dot{x} = f(x(t_i), k(x(t_i), t_i))$.

Definition 3 (π-trajectory). The π-trajectory $x_\pi(.;x_0,k)$ of system (1), associated with a partition $\pi = (t_i)_{i\in\mathbb{N}}$, an initial condition $x_0 = x(t_0)$ and a control strategy k, is the time function obtained by solving successively for every $i \in \mathbb{N}$:

$$\dot{x} = f(x, k(x(t_i),t)) \qquad t \in [0, t_{i+1} - t_i] \qquad i = 0,1,2,\ldots \tag{2}$$

using as initial value the endpoint of the solution of the preceding interval.

Eq. (2) reminds receding horizon. Indeed receding horizon consists in finding, at sampling time t_i, an open-loop control $t \to k(x(t_i), t)$ defined for $t \in [0, T]$ (with $T \geq \bar{\delta}(\pi)$ possibly infinite) and in applying it during the interval $[t_i, t_{i+1}]$. Repeating this scheme gives a control depending upon $x(t_i)$ and the time $t \in [0, t_{i+1} - t_i]$ as in (2). These definitions are a slight generalization of some definitions originally introduced by Clarke et al. in [2] where u was independent of the time.

2.2 Existing result and main contribution

With the above definitions, the result obtained in [2] is the following:

Theorem 2 (Clarke et al. [2]). *Assume that system (1) is globally asymptotically controllable and satisfies Ass. 1. Then, there exists a measurable function* $k : \mathbb{R}^n \to \mathbb{R}^p$ *such that for every real numbers* $R > r > 0$*, there exists* $M(R) > 0$*,* $T(R, r)$ *and* $\delta(R, r) > 0$ *such that for every partition* π *such that* $\bar{d}(\pi) < \delta(R, r)$*, one has:*

1. *(bounded trajectory)* $\forall x_0 \in \mathcal{B}(R),\ \forall t \geq 0,\ x_\pi(t; x_0, k) \in \mathcal{B}(M(R))$.
2. *(attractivity)* $\forall x_0 \in \mathcal{B}(R),\ \forall t \geq T(R, r),\ x_\pi(t; x_0, k) \in \mathcal{B}(r)$.
3. *(stability)* $\lim_{R \to 0} M(R) = 0$.

The above result underlines a relation between global asymptotic stability and a kind of stability, called *s-stability* ("s" stands for sampling) in the original paper [2]. This concept of stabilisation enables the generalization of the concept of stabilisation well known in the continuous case. Indeed, if one takes an initial condition x_0 and a sequence of partitions π_l such that $\bar{d}(\pi_l) \to 0$ as $l \to \infty$, the functions $x_{\pi_l}(.; x_0, k)$ remain in a bounded set. Because $f(x, k(x))$ is also bounded on this set, these functions are equicontinuous hence, using Arzela-Ascoli's Theorem, there is a subsequence that converges to a function that we denote $x(.; x_0, k)$. Any limit $x(.; x_0, k)$ of such convergent subsequences can be considered as a *generalized solution* of the closed loop system $\dot{x} = f(x, k(x))$. These generalized solutions always exists though it may not be unique and the system is globally asymptotically stable with respect to that definition of solution.

Practically, these solutions are impossible calculate and one may prefer to keep the continuous-discrete time scheme, that is to fix a partition. In that case, the obtained stability is clearly not asymptotic since the upper diameter $\bar{d}(\pi)$ of the partition may have to tend to zero with $\delta(R, r)$ as R tends to zero. However, the π-trajectory of the system is guarantied to remain in the ball $\mathcal{B}(r)$ after some time $T(R, r)$, which leads to practical stability of the closed loop system. The aim of this paper is to answer this problem:

Theorem 3. *Assume that system (1) is asymptotically controllable and satisfies Ass. 1, then, for all* $\delta > 0$*, there exists a measurable function* $k : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^p$ *such that:*

1. $\forall x \in \mathbb{R}^n$, $k(x,.) \in \mathcal{L}_\infty^{\mathbb{R}^p}$,
2. $\forall R > r > 0$, *there exists* $M(R) > 0$ *and* $T(R,r) > 0$ *such that for any partition* π *such that* $\underline{d}(\pi) \geq \delta$, *one has:*
   (a) (bounded trajectory) $\forall x_0 \in \mathcal{B}(R), \forall t \geq 0,\ x_\pi(t; x_0, K) \in \mathcal{B}(M(R))$,
   (b) (attractivity) $\forall x_0 \in \mathcal{B}(R),\ \forall t \geq T(R,r),\ x_\pi(t; x_0, K) \in \mathcal{B}(r)$,
   (c) (stability) $\lim_{R\to 0} M(R) = 0$.

Clearly, this theorem has the advantage of insuring the asymptotic stability of the closed-loop system and not only a practical stability, since it is not necessary to sample infinitely fast as the state comes to the origin. On the other hand and like every scheme based on sampling, fixing a priori the sampling schedule may induce problems due to blow-up in finite time. The proposed feedback in its present form is not an exception to this rule. Nevertheless, this can easily be cleared up by making the sampling period δ depend dynamically upon the current state. This point won't be detailed here.

Similarly as in [2], Th. 3 leads to the following definition of global asymptotic *cd*-stability (where "cd" stands for continuous-discrete):

Definition 4. An asymptotically controllable system satisfying Ass. 1 that admits for all $\delta > 0$ a function k as in Th. 3 will be said *globally asymptotically cd-stabilizable*

3 Proof of Theorem 3

The aim of this section is to prove that under Ass. 1 and an asymptotic controllability assumption, one can construct a feedback $k : \mathbb{R}^n \times \mathbb{R}^+$ that asymptotically cd-stabilizes system (1). Let us first begin with the following definition that makes the reading of the sequel easier.

Definition 5 (bounded control strategy). Let denote by bounded control strategy any measurable function $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^p$ such that for all $R > r > 0$, there exists $M(R) > 0$ and $T(R,r) > 0$ satisfying:

1. (stability) $\lim_{R\to 0} M(R) = 0$,
2. (bounded trajectory) $\forall x \in \mathcal{B}(R),\ \forall t \geq 0, \quad x(t; x, u(x,.)) \in \mathcal{B}(M(R))$,
3. (attractivity) $\forall x \in \mathcal{B}(R),\ \forall t \geq T(R,r), \quad x(t; x, u(x,.)) \in \mathcal{B}(r)$.
4. (bounded control) For all compact set $\mathcal{X} \subset \mathbb{R}^n$ and $x \in \mathcal{X}$, $u(x,.)$ belongs almost everywhere to a compact subset $\mathcal{U} \subset \mathbb{R}^p$.

It should be emphasized that this notion concerns the open-loop trajectory contrary to Def. 4 of asymptotic stability where π-trajectories, that is closed-loop trajectories, are considered. This difference is significant since, as underlined in [7], the existence of such open-loop controls is far from implying directly the asymptotic cd-stability. The proof of Th. 3 is based on receding horizon considerations and splits up into the following main parts that make up the three next subsections.

1. Firstly, under Ass. 1 and asymptotic controllability assumption, system (1) is proved to admit a bounded control strategy that enables the definition of a cost function like in the infinite horizon control framework.

2. Then, it will be established that, for every $\delta > 0$, there is a feedback law that decreases this cost function for every partition π such that $\underline{d}(\pi) \geq \delta$.
3. Finally, it will be proved that the above feedback globally asymptotically cd-stabilises system (1).

3.1 Definition of a cost function

The aim of this section is to introduce the following intermediate lemma, which proof is given in Appendix A.

Lemma 1. *If system (1) is asymptotically controllable and satisfies Ass. 1, then there exists a bounded control strategy v, a function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of class $\mathcal{C}^1$ and, for all $R > 0$, a decreasing function $\Lambda_R : \mathbb{R}^+ \to \mathbb{R}^+$ such that:*

1. *G and its derivative g are of class $\mathcal{K}$,*[2]
2. *For every $x \in \mathbb{R}^n$, the below integral $W(x, v(x, .))$ converges*

$$W(x, v(x, .)) := \int_0^{+\infty} G\left(\|x(\tau; x, v(x, .))\|\right) d\tau \tag{3}$$

3. *$R_1 \geq R_2 \geq 0 \quad \Rightarrow \quad \Lambda_{R_1}(0) \geq \Lambda_{R_2}(0)$,*
4. *$\lim_{R\to 0} \Lambda_R(0) = 0$,*
5. *$\forall x \in \mathcal{B}(R)$, $\forall t \geq 0$, $\|x(t; x, v(x, .))\| \leq \Lambda_R(t)$,*
6. *$\int_0^{+\infty} G(\Lambda_R(\tau))d\tau$ converges.*

Clearly, items 5 and 6 imply item 2. $W(x, v(x, .))$ is the cost associated with the initial state x and the open-loop control $v(x, .)$. It takes the receding horizon classical form when no weighting is put on the control.

The proof of this lemma, detailed in Appendix A, splits up into in the following points:

1. In a first step, it is proved that system (1) admits a bounded control strategy v as soon as it is asymptotically controllable.
2. This enables to define, for all $R > 0$, a *"gauge"* function Λ_R, fulfilling items 3, 4 and 5 of Lem. 1.
3. Fixing $\bar{R} = R$ and using Massera's lemma [8, Lem. 12], gives a function G fulfilling item 1 of Lem. 1 and such that $\int_0^{+\infty} G(\Lambda_{\bar{R}}(\tau))d\tau$ converges.
4. Verifying that $\int_0^{+\infty} G(\Lambda_R(\tau))d\tau$ converges for every R ends the proof.

3.2 Formulation of the feedback

Let $C > 0$ be a real constant and $\mathcal{B}(\frac{C}{2^k})$, $k \in \mathbb{Z}$, be concentric balls defining a subdivision of $\mathbb{R}^n$. It follows from the asymptotic controllability assumption and from Lem. 1, that there exists for every $k \in \mathbb{Z}$, a compact set $\mathcal{U}_{\frac{C}{2^k}}$ such that for all $x \in \mathcal{B}(\frac{C}{2^k})$, $v(x, t)$ belongs almost everywhere to $\mathcal{U}_{\frac{C}{2^k}}$. For all $x \in \mathbb{R}^n \backslash \{0\}$, let us define:

[2] following Hahn [5], any continuous strictly increasing functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(0) = 0$ will be said of class $\mathcal{K}$

- $n_x \in \mathbb{Z}$ be the larger relative integer n such that $x \in \mathcal{B}(\frac{C}{2^n})$. According to Lem. 1 and for all $x \in \mathbb{R}^n$, one has:

$$\forall t \geq 0, \qquad \|x(t;x,v(x,.))\| \leq \Lambda_{\frac{C}{2^{n_x}}}(t)$$

with in addition, $v(x,t) \in \mathcal{U}_{\frac{C}{2^{n_x}}}$ for almost every $t \geq 0$.

- $\bar{n}_x \in \mathbb{Z}$ be the larger relative integer n such that for which there exists an open-loop control $u : \mathbb{R}^+ \to \mathbb{R}^p$ such that:

$$\begin{cases} u(t) \in \mathcal{U}_{\frac{C}{2^{\bar{n}_x}}} \text{ almost everywhere} \\ \forall t \geq 0, \quad \|x(t;x,u)\| \leq \Lambda_{\frac{C}{2^{\bar{n}_x}}}(t) \end{cases} \tag{4}$$

According to the previous item, it is clear that:

$$\bar{n}_x \geq n_x \tag{5}$$

Note also that along every open-loop trajectory $x(.;x,u)$, one has:

$$\bar{n}_{x(t;x,u)} \geq \bar{n}_x \qquad \forall t \geq 0 \tag{6}$$

Indeed, $u(.+t)$ is an open-loop control belonging to $\mathcal{U}_{\frac{C}{2^{\bar{n}_x}}}$. Hence, using (4) and the decrease of $\Lambda_{\frac{C}{2^{\bar{n}_x}}}$, one has for every $t' \geq 0$:

$$\|x(t';x(t;x,u),u(.+t))\| = \|x(t'+t;x,u)\| \leq \Lambda_{\frac{C}{2^{\bar{n}_x}}}(t'+t) \leq \Lambda_{\frac{C}{2^{\bar{n}_x}}}(t')$$

- $U_x \subset \mathcal{L}_\infty^{\mathbb{R}^p}$ denote the set[3] of open-loop control u fulfilling conditions (4).

Finally, let $W(x)$ be the minimum cost associated with x[4]:

$$W(x) := \inf \{W(x,u);\ u \in U_x\} \tag{7}$$

Lemma 2. *For all $\delta > 0$, there is a definite function $\varepsilon_\delta(x)$ (that is $\varepsilon_\delta(x) = 0 \Leftrightarrow x = 0$), such that every feedback of the form (8) cd-stabilise (1).*

$$K(x,t) := u(t) \quad \textit{with } u \in V_x := \{u \in U_x;\ W(x,u) \leq W(x) + \varepsilon_\delta(x)\} \tag{8}$$

The proof of this last lemma ends the proof of Th. 3.

3.3 Asymptotic cd-stability of the closed-loop system

First of all, note that the definition of U_x and (6) give:

$$\forall t \geq 0, \qquad \bar{n}_{x_\pi(t;x,K)} \geq \bar{n}_x \qquad \text{along the } \pi\text{-trajectories} \tag{9}$$

Items 1, 2(a) and 2(c) of Th. 3 are quite easy to verify:

1. By (8), for all $x \in \mathbb{R}^n$, one has $K(x,.) = u(.) \in V_x \subset U_x \subset \mathcal{L}_\infty^{\mathbb{R}^p}$.

---

[3] by definition of $\bar{n}_x$, the set U_x can not be empty.

[4] it does not necessary exist a control $u \in \mathcal{L}_\infty^{\mathbb{R}^p}$ such that $W(x,u) = W(x)$.

2. For every partition $\pi = (t_i)_{i\in\mathbb{N}}$, the π-trajectory of (1) with feedback (8) satisfies:
   (a) for all $R > 0$, all $x \in \mathcal{B}(R)$ and all $t \geq 0$ (since $\Lambda_{\frac{C}{2^{n_x}}}$ is decreasing):
   $$x_\pi(t;x,K) \underset{(9)}{\in} \mathcal{B}\left(\sup_{t\in\mathbb{R}^+} \Lambda_{\frac{C}{2^{n_x}}}(t)\right) \subset \mathcal{B}\left(\Lambda_{\frac{C}{2^{n_x}}}(0)\right) \underset{\substack{\text{Lemma 1.3}\\ \text{and (5)}}}{\subset} \mathcal{B}\left(\Lambda_{\frac{C}{2^{\bar{n}_x}}}(0)\right)$$
   Hence, one has $x_\pi(t;x,K) \in \mathcal{B}(\Lambda_{\frac{C}{2^{n_R}}}(0))$ with $n_R := \inf_{x\in\mathcal{B}(R)} n_x$.
   (c) Lemma 1.4 gives: $\lim_{R\to 0} \Lambda_{\frac{C}{2^{n_R}}}(0) = 0$

In order to conclude, it only remains to prove item 2(b) of Th. 3. Let $R > r > 0$ be two real numbers, $x \in \mathcal{D}(r,R)$[5] be the initial state of system (1) and $x_\pi(t_i;x,K)$ the state of du system (1) at time t_i of the partition, when the feedback K defined by (8) is applied. The aim is to prove that there exists a time $T(r,R)$ such that for every partition π of lower diameter $\underline{d}(\pi) \geq \delta$ and every $x \in \mathcal{D}(r,R)$, one has $x_\pi(t;x,K) \in \mathcal{B}(r)$, for all $t \geq T(r,R)$. This proof follows the three following steps:

1. First of all, it will be proved that for all $R > r > 0$, there is an integer $N_1(r,R)$ such that, for all $x \in \mathbb{R}^n$, for every partition π, such that $\underline{d}(\pi) \geq \delta$, and for all instant t_i of it such that $x_\pi(t_i;x,K) \in \mathcal{D}(r,R)$, one has $\bar{n}_{x_\pi(t_{i+N_1(r,R)};x,K)} > \bar{n}_{x_\pi(t_i;x,K)}$. In other words, $\bar{n}_{x_\pi(t_i;x,K)}$ increments of one, at worst, every $N_1(r,R)$ sampling period. This directly follows from the choice of ε_δ that will be done in the following and that will insure the decrease of the cost function W at each sampling time t_i. This will imply the decrease of $\|x_\pi(t_i;x,K)\|$ and hence, after some steps, the growth of $\bar{n}_{x_\pi(t_i;x,K)}$.
2. It will follow quite easily from the previous item that for all $R > r > 0$, there exists an integer $N(r,R)$ such that for every partition of lower diameter greater than δ and every $x \in \mathcal{D}(r,R)$, one has $x_\pi(t;x,K) \in \mathcal{B}(r)$, for all $t \geq t_{N(r,R)}$. This last point is almost the objective of the present section with this slight difference that $N(r,R)$ does not depend upon the partition contrary to $t_{N(r,R)}$.
3. Finally, the existence of a time $T(r,R)$, independent of the partitions will be proved. This last point follows from the choice of the open-loop controls maid such that the corresponding trajectories remain below the gauge function $\Lambda_R(t)$. If for some i, $t_{i+1} - t_i$ happens to be to large, the trajectory will naturally reach the ball $\mathcal{B}(r)$ *"in open-loop"*. Hence, it can be deduced that there is a time $T(r,R)$ after which, even if $t_{N(r,R)} \geq T(r,R)$, the trajectory will reach the ball $\mathcal{B}(r)$.

Existence of $N_1(r,R)$ Let $\delta > 0$ be a fixed real number and Γ denote the primitive of G vanishing at the origin. From (9) and for all partition π, one has along the trajectories: $\bar{n}_{x_\pi(t_{i+1};x,K)} \geq \bar{n}_{x_\pi(t_i;x,K)}$. For the ease of the

[5] where $\mathcal{D}(r,R)$ denotes the closed disk of lower radius r and upper radius R

reader, let $x_i := x_\pi(t_i; x, K)$ denote the state of system (1) at time t_i. At time t_{i+1}, the state x_{i+1} of the system is given by integrating system (1) between t_i and t_{i+1} with control $K(x_i, t) = u_i(t)$ as defined by (8). Hence:

$$W(x_i) \le W(x_i, u_i) \le W(x_i) + \varepsilon_\delta(x_i)$$

Now, if $\bar{n}_{x_{i+1}} = \bar{n}_{x_i}$, $u_i(. - (t_{i+1} - t_i))$ also belongs to $U_{x_{i+1}}$, giving:

$$\begin{aligned} W(x_{i+1}) &\le W(x_{i+1}, u_i(. - (t_{i+1} - t_i))) \\ &\le W(x_i, u_i) - \int_0^{t_{i+1}-t_i} G(\|x(\tau; x_i, u_i)\|) d\tau \\ &\le W(x_i) + \varepsilon_\delta(x_i) - \int_0^{t_{i+1}-t_i} G(\|x(\tau; x_i, u_i)\|) d\tau \end{aligned} \tag{10}$$

Using $t_{i+1} - t_i \ge \delta$ and defining[6]:

$$\forall \rho > 0, \qquad S(\rho) := \max(\sup_{\substack{x \in \mathcal{B}(\Lambda_\rho(0)) \\ u \in \mathcal{U}_\rho}} \|f(x, u)\|, 1) \tag{11}$$

it follows that[7]:

$$\begin{aligned} \int_0^{t_{i+1}-t_i} G(\|x(\tau; x_i, u_i)\|) d\tau &\ge \int_0^{\min\left(\frac{\|x_i\|}{S\left(\frac{C}{2^{\bar{n}_{x_i}}}\right)}, \delta\right)} G\left(\|x_i\| - S\left(\frac{C}{2^{\bar{n}_{x_i}}}\right)\tau\right) d\tau \\ &\ge \frac{1}{S(\frac{C}{2^{\bar{n}_{x_i}}})} \left[\Gamma(\|x_i\|) - \Gamma\left(\|x_i\| - \min\left(\|x_i\|, S\left(\frac{C}{2^{\bar{n}_{x_i}}}\right)\delta\right)\right)\right] \end{aligned} \tag{12}$$

Combining inequalities (10) and (12) gives:

$$W(x_{i+1}) - W(x_i) \le \varepsilon_\delta(x_i) - \frac{\Gamma(\|x_i\|) - \Gamma\left(\|x_i\| - \min\left(\|x_i\|, S\left(\frac{C}{2^{\bar{n}_{x_i}}}\right)\delta\right)\right)}{S(\frac{C}{2^{\bar{n}_{x_i}}})}$$

Hence, the aim here consist in doing an appropriate choice of ε_δ in order to force the right member of this last inequality to remain strictly negative. Let:

$$\varepsilon_\delta(x) := \min\left(\frac{1}{S(\frac{C}{2^{\bar{n}_x}})}, \frac{\delta}{\|x\|}\right) \frac{\Gamma(\|x\|)}{2} \tag{13}$$

With such a choice:

- if $\|x_i\| \le S(\frac{C}{2^{\bar{n}_{x_i}}})\delta$, one has:

$$W(x_{i+1}) - W(x_i) \le \varepsilon_\delta(x_i) - \frac{1}{S(\frac{C}{2^{\bar{n}_{x_i}}})}\Gamma(\|x_i\|) \underset{(13)}{\le} -\varepsilon_\delta(x_i) \tag{14}$$

---

[6] $S(\frac{C}{2^{\bar{n}_x}})$ is an upper bound to the time derivative of $x(.; x, u)$ when $u \in U_x$

[7] there is a "min" function in this expression since $\tau \to \|x_i\| - S(\frac{C}{2^{\bar{n}_{x_i}}})\tau$ vanishes before $\tau = \delta$ if $\frac{\|x_i\|}{S(\frac{C}{2^{\bar{n}_{x_i}}})} \le \delta$

- if $\|x_i\| > S(\frac{C}{2^{\bar{n}_{x_i}}})\delta$, one has using (13):

$$W(x_{i+1}) - W(x_i) \leq \frac{\delta\Gamma(\|x_i\|)}{2\|x_i\|} - \frac{\Gamma(\|x_i\|) - \Gamma\left(\|x_i\| - \delta S\left(\frac{C}{2^{\bar{n}_{x_i}}}\right)\right)}{S\left(\frac{C}{2^{\bar{n}_{x_i}}}\right)}$$

$$\leq -\varepsilon_\delta(x_i) - \frac{1}{S\left(\frac{C}{2^{\bar{n}_{x_i}}}\right)}\left[\left(1 - \frac{\delta S(\frac{C}{2^{\bar{n}_{x_i}}})}{\|x_i\|}\right)\Gamma(\|x_i\|) - \Gamma\left(\|x_i\| - \delta S(\frac{C}{2^{\bar{n}_{x_i}}})\right)\right]$$

Since $0 < \frac{\delta S(\frac{C}{2^{\bar{n}_{x_i}}})}{\|x_i\|} < 1$ and, G being increasing, its primitive Γ is convex, the second term of this inequality is negative or null. Hence:

$$W(x_{i+1}) - W(x_i) \leq -\varepsilon_\delta(x_i) \tag{15}$$

In both cases (14) and (15), one has $W(x_{i+1}) - W(x_i) \leq -\varepsilon_\delta(x_i)$. Moreover, defining for any $R > r > 0$, $\underline{\varepsilon}_\delta(r, R) := \inf_{x\in\mathcal{D}(r,R)} \varepsilon_\delta(x)$, it can be proved that $\underline{\varepsilon}_\delta(r, R) > 0$. Indeed, by (11), one necessarily has $S(\frac{C}{2^{\bar{n}_x}}) \leq S(\frac{C}{2^{n_x}}) \leq S(\frac{C}{2^{n_R}})$, with $n_R := \inf_{x\in\mathcal{B}(R)} n_x$. Therefore, with the definition of ε_δ, it follows:

$$\varepsilon_\delta(x) = \min\left(\frac{1}{S(\frac{C}{2^{\bar{n}_x}})}, \frac{\delta}{\|x\|}\right)\frac{\Gamma(\|x\|)}{2} \geq \min\left(\frac{1}{S\left(\frac{C}{2^{n_R}}\right)}, \frac{\delta}{\|x\|}\right)\frac{\Gamma(\|x\|)}{2} \tag{16}$$

The right member of inequality (16) is clearly continuous with respect to x and hence one effectively has $\underline{\varepsilon}_\delta(r, R) > 0$, for all $R \geq r > 0$.

Remark 1. δ can be chosen as small as needed, however, it can not taken null. Indeed, it would be then impossible to insure that $\underline{\varepsilon}_\delta(r, R) > 0$ which precisely gives the convergence of the state to the origin.

With the two points below illustrated on Fig. 1, one can define $N_1(r, R)$:

- According to Lem. 1, every trajectory starting in the ball $\mathcal{B}(R)$ remains in the ball $\mathcal{B}(\Lambda_R(t))$ for all $t \geq 0$. In particular, one has:

$$x \in \mathcal{D}(r, R) \Rightarrow W(x) \leq \int_0^{+\infty} G(\Lambda_R(\tau))d\tau \tag{17}$$

- According to (11), the time derivative of the trajectory can be bounded on every compact set. Hence if $x \in \mathcal{D}(r, R)$, the cost $W(x)$ will necessarily be larger than a minimum cost $W_{min}(r)$ corresponding to the fastest decrease of the state:

$$W(x) < W_{min}(r) \Rightarrow x \in \mathcal{B}(r) \tag{18}$$

Consequently, choosing $N_1(r, R)$ as the smallest integer such that:

$$N_1(r, R) > \frac{1}{\underline{\varepsilon}(r, R)}\left(\int_0^{+\infty} G(\Lambda_R(\tau))d\tau - W_{min}(r)\right) \tag{19}$$

For all $x \in \mathcal{D}(r, R)$, one has the following relation that ends the proof of the first points described above:

$$\bar{n}_{x_\pi\left(t_{i+N_1(r,R)};x,K\right)} > \bar{n}_{x_\pi(t_i;x,K)} \tag{20}$$

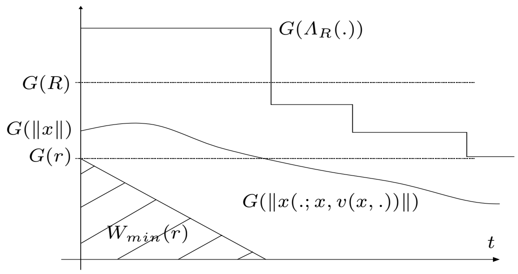

Fig. 1. Upper and lower bound of $W(x)$ for $x \in \mathcal{D}(r, R)$

Existence de $N(r, R)$ The existence of N directly follows from the one of N_1. Indeed, let $m(r) := \inf\{m \in \mathbb{Z};\ \mathcal{B}(M(\frac{C}{2^m})) \subset \mathcal{B}(r)\}$. Clearly, $m(r)$ is the smallest integer such that, for all $x \in \mathcal{B}(\frac{C}{2^{m(r)}})$, the π-trajectory $x_\pi(.; x, K)$ remains in $\mathcal{B}(r)$. Using this and since, for any $x \in \mathcal{D}(r, R)$, one has $n_x \geq n_R$, one can easily verify that, choosing for all $R > r > 0$, $N(r, R) := N_1(\frac{C}{2^{m(r)}}, R)(m(r) - n_R)$ guarantees that, for any partition π of lower diameter $\underline{d}(\pi) \geq \delta$, any $x \in \mathcal{D}(r, R)$ and any $i \geq N(r, R)$, one has $\bar{n}_{x_\pi(t_i;x,K)} < m(r)$. Therefore, for any $i \geq N(r, R)$, $x_\pi(t_i; x, K) \in \mathcal{B}(\frac{C}{m(r)})$, giving for all $t \geq t_{N(r,R)}$, $x_\pi(t; x, K) \in \mathcal{B}(M(\frac{C}{m(r)})) \subset \mathcal{B}(r)$. This is exactly our second objective.

Existence of $T(r, R)$ It only remains to prove that $T(r, R)$ can be chosen independently of the partition to conclude the proof of Th. 3. First of all, recall that, in the continuous-discrete time scheme used here (see Def. 3), the system evolves in open-loop between two sampling instants. According to (8), the control u is chosen at each sampling time $(t_i)_{i \in [0, N(r,R)]}$ in the set $V_{x_\pi(t_i;x,K)} \subset U_{x_\pi(t_i;x,K)}$. According to (6), $u \in U_x$, and hence using (4), one has: $x(t; x, u) \in \mathcal{B}(\Lambda_R(t))$. Consequently, if the time between a sampling instant t_i and the next one t_{i+1} becomes *too large*, the trajectory will meet in open-loop the ball $\mathcal{B}(m(r))$ in a time less than $T_{max}(r, R)$. Since $T_{max}(r, R)$ depends only upon r and R one can conclude by taking $T(r, R) := N(r, R) T_{max}(r, R)$. That way, for any partition π of lower diameter $\underline{d}(\pi) \geq \delta$ and for any $x \in \mathcal{D}(r, R)$, one has: $\forall t \geq T(r, R), \quad x_\pi(t; x, K) \in \mathcal{B}(r)$. This last point ends the proof of Th. 3.

A Proof of Lemma 1

Recall that the following proof is made up with four main points detailed in section 3.1.

Existence of v According to Th. 2, there exists a feedback law κ such that, for any $R > r > 0$, there is $M(R) > 0$, $T(R, r) > 0$, $\delta(R, r)$ and a partition $\pi(R, r)$ of upper diameter $\bar{d}(\pi) \leq \delta(R, r)$, so that:

- (bounded trajectory) $\forall x \in \mathcal{B}(R)$, $\forall t \geq 0$, $\quad x_{\pi(R,r)}(t;x,\kappa) \in \mathcal{B}(M(R))$,
- (attractivity) $\forall x \in \mathcal{B}(R)$, $\forall t \geq T(R,r)$, $\quad x_{\pi(R,r)}(t;x,\kappa) \in \mathcal{B}(r)$,
- (stability) $\lim_{R\to 0} M(R) = 0$.

Moreover, for any compact $\mathcal{X}$ of $\mathbb{R}^n$ and all $x \in \mathcal{X}$, $\kappa(x_{\pi(R,r)}(t;x,\kappa))$ is in a compact subset $\mathcal{U}_{\rho(\mathcal{X})}$ of $\mathbb{R}^p$ depending only upon $\mathcal{X}$ [2].

In order to simplify the notations, let $\pi_x := \pi(\|x\|, \frac{\|x\|}{2})$ be a partition such that $\bar{d}(\pi_x) \leq \delta(\|x\|, \frac{\|x\|}{2})$ and let the control strategy w be defined by:

$$w(x,t) := \kappa\left(x_{\pi_x}(t_i;x,\kappa)\right) \qquad t \in [t_i, t_{i+1}] \tag{21}$$

For all $x \in \mathbb{R}^n$, one has $x(t;x,w(x,.)) = x_{\pi_x}(t;x,\kappa)$ for all $t \geq 0$. The open-loop trajectory[8] $x(t;x,w(x,.))$ obtained by applying the control law w is clearly identical to the π-trajectory $x_{\pi_x}(t;x,\kappa)$. Hence, the system will meet the ball $\mathcal{B}(\frac{\|x\|}{2})$ in a time less than $T(\|x\|, \frac{\|x\|}{2})$. In order to obtain a bounded control strategy in the sense of Def. 5, it only remains to prove the attractivity of the origin. This can be simply obtained by applying *repetitively* the control strategy w. For all $x \in \mathbb{R}^n$, let:

$$\begin{aligned} x_0 := x \qquad & x_{k+1} := x\left(t_k(\|x\|); x_k, w(x_k,.)\right) \\ \text{with:} \quad & t_k(\|x\|) := T\left(\tfrac{\|x\|}{2^k}, \tfrac{\|x\|}{2^{k+1}}\right) \end{aligned} \tag{22}$$

$t_k(\|x\|)$ is the time[9] needed to go from a state of norm $\frac{\|x\|}{2^k}$ to a state of norm less than $\frac{\|x\|}{2^{k+1}}$. For all $t \geq 0$ and $R > 0$, let k_R^t and T_R^t be defined by:

$$k_R^t := \begin{cases} 0 & \text{if } t \leq t_0(R) \\ \text{the unique integer } k \text{ such that:} & \\ \quad t \in \left]\sum_{j=0}^{k} t_j(R), \sum_{j=0}^{k+1} t_j(R)\right] & \text{if } t > t_0(R) \end{cases} \tag{23}$$

$$T_R^t := \begin{cases} 0 & \text{if } t \leq t_0(R) \\ \sum_{j=0}^{k_R^t} t_j(R) & \text{if } t > t_0(R) \end{cases} \tag{24}$$

and v be given by:

$$v(x,t) := \begin{cases} w(x,t) & \text{for } t \leq t_0(\|x\|) \\ w\left(x\left(T_{\|x\|}^t; x_{k_{\|x\|}^t}, v(x_{k_{\|x\|}^t},.)\right), t - T_{\|x\|}^t\right) & \text{for } t > t_0(\|x\|) \end{cases} \tag{25}$$

For all $x \in \mathbb{R}^n$, the open-loop control $v(x,.)$ gives the generic trajectory profile $x(.;x,v(x,.))$ depicted on Fig. 2.

It is then easy to verify that v is a bounded control strategy in the sense of Def. 5. This gives the first point of the proof: for all $R > r > 0$, there is $M_v(R) := M(R) > 0$ and $T_v(R,r)$ so that:

[8] $w(x,.)$ is an open-loop control strategy independent of any partition, though it is deduced from a partition π_x.

[9] $t_k(\|x\|)$ has nothing to do with any partition

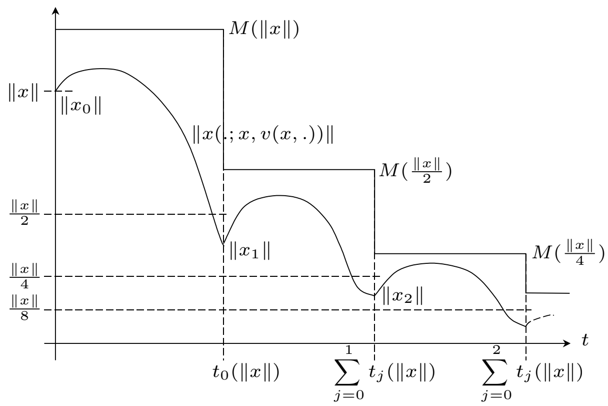

Fig. 2. Generic state trajectory $x(.; x, v(x, .))$

1. $\lim_{R\to 0} M_v(R) = \lim_{R\to 0} M(R) = 0$,
2. $\forall x \in \mathcal{B}(R),\ \forall t \geq 0,\ x(t; x, v(x, .)) \in \mathcal{B}(M_v(R)) = \mathcal{B}(M(R))$,
3. Let n_R be the smallest integer such that $M(\frac{R}{2^{n_R}}) \leq r$ and, for all $x \in \mathcal{B}(R)\backslash\mathcal{B}(\frac{R}{2^{n_R}})$, let n_x be the smallest integer[10] such that $M(\frac{\|x\|}{2^{n_x}}) \leq r$. Then, for all $t \geq \sum_{j=0}^{n_x-1} t_j(\|x\|)$, one has $x(t; x, v(x, .)) \in \mathcal{B}(M(\frac{\|x\|}{2^{n_x}})) \subset \mathcal{B}(r)$. Noticing that every trajectory with initial condition in $\mathcal{B}(\frac{R}{2^{n_R}})$ remains in $\mathcal{B}(r)$, it becomes clear that it is sufficient to bound $\sum_{j=0}^{n_x-1} t_j(\|x\|)$ for $x \in \mathcal{B}(R)\backslash\mathcal{B}(\frac{R}{2^{n_R}})$ in order to conclude:

$$\sum_{j=0}^{n_x-1} t_j(\|x\|) \underset{(22)}{=} \sum_{j=0}^{n_x-1} T\left(\frac{\|x\|}{2^j}, \frac{\|x\|}{2^{j+1}}\right) \leq (n_x - 1)T\left(R, \frac{R}{2^{n_R+1}}\right)$$

$$\leq (n_R - 1)\, T\left(R, \frac{R}{2^{n_R+1}}\right) =: T_v(r, R) \quad (26)$$

   Hence, for all $x \in \mathcal{B}(R)$ and $t \geq T_v(r, R)$, one has $x(t; x, v(x, .)) \in \mathcal{B}(r)$[11]
4. For any compact set $\mathcal{X} \subset \mathbb{R}^n$, the open-loop control $v(x, .)$ satisfying the two previous points is in a compact subset $\mathcal{U} \subset \mathbb{R}^p$ almost everywhere since κ also satisfies this property (see [2] for further details on the construction of κ).

Definition of Λ_R Figure 3 illustrates the construction that follows. Let $\bar{R} > 0$, be a fixed radius. Let $\lambda_{\bar{R}} : \mathbb{R}^+ \to \mathbb{R}^+$ be the above defined function:

$$\lambda_{\bar{R}}(t) := \begin{cases} M_v(\bar{R}) & \text{if } t \in \left[0, T_v\left(\bar{R}, \frac{\bar{R}}{2}\right)\right] \\ M_v\left(\frac{\bar{R}}{2^k}\right) & \text{if } t \in \left]T_v\left(\bar{R}, \frac{\bar{R}}{2^k}\right), T_v\left(\bar{R}, \frac{\bar{R}}{2^{k+1}}\right)\right] \end{cases} \quad (27)$$

[10] note that $x \in \mathcal{B}(R)$ implies that $n_x \leq n_R$.
[11] Note that $T_v(r, R)$ depends upon r through n_R

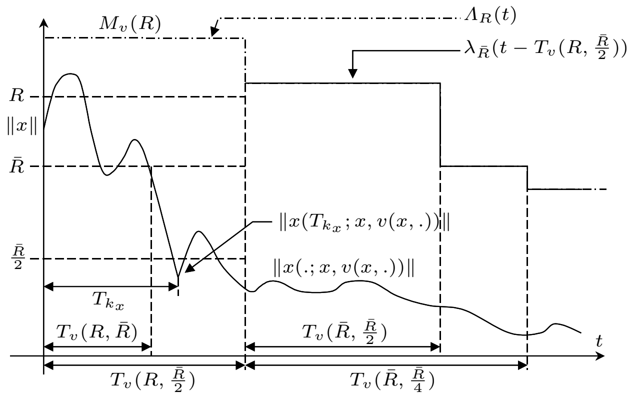

Fig. 3. Illustration of function Λ_R

Since v is a bounded control strategy, for all $x \in \mathcal{B}(\bar{R})$ and $t \geq 0$, one has $x(t; x, v(x, .)) \in \mathcal{B}(\lambda_{\bar{R}}(t))$. Function $\Lambda_R : \mathbb{R}^+ \to \mathbb{R}^+$ defined below extends this result for all $R > 0$ and $x \in \mathcal{B}(R)$:

$$\Lambda_R(t) := \begin{cases} M_v(R) & \text{if } R > \bar{R} \text{ and } t < T_v(R, \frac{\bar{R}}{2}) \\ \lambda_{\bar{R}}(t - T_v(R, \frac{\bar{R}}{2})) & \text{if } R > \bar{R} \text{ and } t \geq T_v(R, \frac{\bar{R}}{2}) \\ \min\left(M_v(R), \lambda_{\bar{R}}(t)\right) & \text{if } R \leq \bar{R} \end{cases}$$

For all $R \leq \bar{R}$ and $x \in \mathcal{B}(R)$, one knows that for all $t \geq 0$, $x(t; x, v(x, .)) \in \mathcal{B}(\lambda_{\bar{R}}(t))$. By construction of v, one also knows that $x(t; x, v(x, .)) \in M_v(R)$. These two points give that for all $R \leq \bar{R}$, all $x \in \mathcal{B}(R)$ and all $t \geq 0$, one has $x(t; x, v(x, .)) \in \mathcal{B}(\Lambda_R(t))$.

For all $R > \bar{R}$ and $x \in \mathcal{D}(R, \bar{R})$, there is an integer k_x such that $T_{k_x} := \sum_{j=0}^{k_x} t_j(\|x\|) \in [T_v(R, \bar{R}), T_v(R, \frac{\bar{R}}{2})]$ and $x(T_{k_x}; x, v(x, .)) \in \mathcal{B}(\bar{R})$ (with $t_k(\|x\|)$ defined by (22)). Now, by definition of v, one has for all $t \geq 0$:

$$\|x\left(t; x\left(T_{k_x}; x, v(x, .)\right), v\left(x(T_{k_x}; x, v(x, .)), .\right)\right)\| = \|x\left(t + T_{k_x}; x, v(x, .)\right)\|$$

hence, for all $t \geq T_v(R, \frac{\bar{R}}{2})$, one has using (27) and the decrease of $\lambda_{\bar{R}}$:

$$\|x\left(t; x, v(x, .)\right)\| \leq \lambda_{\bar{R}}(t - T_{k_x}) \leq \lambda_{\bar{R}}\left(t - T_v\left(R, \frac{\bar{R}}{2}\right)\right) \tag{28}$$

Using (28) together with the fact that for all $x \in \mathcal{B}(R)$ and all $t \geq 0$, $x(t; x, v(x, .)) \in B(M_v(R))$, one gets item 5 of Lem. 1, namely, for every $R > 0$, every $x \in \mathcal{B}(R)$ and every $t \geq 0$, one has $x(t; x, v(x, .)) \in \mathcal{B}(\Lambda_R(t))$.

The decrease of function Λ_R and item 3 of Lem. 1 are clear. Item 4 directly follows from the construction of Λ_R: $\Lambda_R(0) \leq M_v(R)$ with $\lim_{R \to 0} M_v(R) = 0$.

Obtaining *G* $\lambda_{\bar{R}}$ is a strictly positive function such that for every $t \geq 0$, $\lim_{t \to \infty} \lambda_{\bar{R}}(t) = 0$ (recall $\lim_{R \to 0} M(R) = 0$). Using Massera's lemma [8, Lem. 12], there is a function G of class $\mathcal{K}$ with derivative g also of class $\mathcal{K}$ such that $\int_0^{+\infty} G\left(\lambda_{\bar{R}}(\tau)\right) d\tau$ converges.

Convergence of $\int_0^\infty G(\Lambda_R(\tau))d\tau$ In order to conclude, it only remains to verify that $\int_0^\infty G(\Lambda_R(\tau))d\tau$ is convergent *for all* $R > 0$. This follows quite easily from the convergence of $\int_0^{+\infty} G\left(\lambda_{\bar{R}}(\tau)\right) d\tau$. For $R \le \bar{R}$, $\Lambda_R(t) = \lambda_{\bar{R}}(t)$, and hence $\int_0^{+\infty} G\left(\Lambda_R(\tau)\right) d\tau$ is convergent. For $R > \bar{R}$, one has:

$$\begin{aligned}\int_0^{+\infty} G\left(\Lambda_R(\tau)\right) d\tau &= \int_0^{T_v(R,\frac{\bar{R}}{2})} G\left(\Lambda_R(\tau)\right) d\tau + \int_{T_v(R,\frac{\bar{R}}{2})}^{+\infty} G\left(\Lambda_R(\tau)\right) d\tau \\ &= \int_0^{T_v(R,\frac{\bar{R}}{2})} G\left(\Lambda_R(\tau)\right) d\tau + \int_0^{+\infty} G\left(\lambda_{\bar{R}}(\tau)\right) d\tau\end{aligned}$$

$\int_0^\infty G(\Lambda_R(\tau))d\tau$ is also convergent. This ends the proof of Lem. 1.

References

1. Brockett, R. W., Millman, R. S., and Susmann, H. S. (1983) Asymptotic stability and feedback stabilization. In: Differential Geometric Control Theory. Birkhäuser, Boston-Basel-Stuttgart.
2. Clarke, F. H., Ledyaev, Y. S., Sontag, E. D., and Subbotin, A. (1997) Asymptotic controllability implies feedback stabilization. IEEE Trans. on Automatic Control, **42**(10):1394–1407.
3. Coron, J. M. and Rosier, L. (1994) A relation between continuous time varying and discontinuous feedback stabilization. Journal of Mathematical Systems, Estimation and Control, **4**(1):64–84.
4. Filippov, A. F. (1988) Differential equations with discontinuous righthand sides. Kluwer Academic Publishers, Dordrecht, Boston, London.
5. Hahn, W. (1967) Stability of motion. Springer Verlag, Berlin-Heidelberg.
6. Kawski, M. (1990) Stabilization of nonlinear systems in the plane. Systems & Control Letters, **12**(2):169–175.
7. Marchand, N. (2000) Commande à horizon fuyant : théorie et mise en œuvre. PhD Thesis, Lab. d'Automatique - INPG, Grenoble, France.
8. Massera, J. L. (1949) On Liapounoff's conditions of stability. Annals of Mathematics, **50**(3):705–721.
9. Sontag, E. D. (1983) A Lyapunov-like characterization of asymptotic controllability. Siam Journal on Control and Optimization, **21**:462–471.
10. Ryan, E. P. (1994) On Brockett's condition for smooth stabilizability and its necessity in a context of nonsmooth feedback. Siam Journal on Control and Optimization, **32**(6):1597–1604.
11. Sontag, E. D. and Sussmann, H. J. (1995) Nonsmooth control-Lyapunov functions. In: Proc. of the IEEE conf. on Decision and Control. New Orleans, USA, 2799–2805.
12. Sontag, E. D. (1998) Mathematical control theory, deterministic finite dimensional systems. Springler Verlag, New York Berlin Heidelberg, second edition.
13. Sontag, E. D. (1999) Stability and stabilization: Discontinuities and the effect of disturbances. Nonlinear Analysis, Differential Equations, and Control. Kluwer. 551–598.
14. Zabczyk, J. (1989) Some comments on stabilizability. Appl. Math. Optim., **19**(1):1–9.